THE SPREAD OF A RUMOR OR INFECTION IN A MOVING POPULATION

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Abstract. We consider the following interacting particle system: There is a "gas" of particles, each of which performs a continuous time simple random walk on $\mathbb{Z}^d$, with jumprate D_A. These particles are called A-particles and move independently of each other. They are regarded as individuals who are ignorant of a rumor or are healthy. We assume that we start the system with $N_A(x, 0-)$ A-particles at x, and that the $N_A(x, 0-)$, $x \in \mathbb{Z}^d$, are i.i.d., mean μ_A Poisson random variables. In addition, there are B-particles which perform continuous time simple random walks with jumprate D_B. We start with a finite number of B-particles in the system at time 0. B-particles are interpreted as individuals who have heard a certain rumor or who are infected. The B-particles move independently of each other. The only interaction is that when a B-particle and an A-particle coincide, the latter instantaneously turns into a B-particle.

We investigate how fast the rumor, or infection, spreads. Specifically, if $\widetilde{B}(t) := \{x \in \mathbb{Z}^d : \text{a } B\text{-particle visits } x \text{ during } [0,t]\}$, and $B(t) = \widetilde{B}(t) + [-1/2, 1/2]^d$, then we investigate the asymptotic behavior of $B(t)$. Our principal result states that if $D_A = D_B$ (so that the A and B-particles perform the same random walk), then there exist constants $0 < C_i < \infty$ such that almost surely $\mathcal{C}(C_2 t) \subset B(t) \subset \mathcal{C}(C_1 t)$ for all large t, where $\mathcal{C}(r) = [-r, r]^d$. In a further paper we plan to prove a full "shape theorem", saying that $t^{-1}B(t)$ converges almost surely to a nonrandom set B_0, with the origin as an interior point, so that the true growth rate for $B(t)$ is linear in t.

If $D_A \neq D_B$, then we can only prove the upper bound $B(t) \subset \mathcal{C}(C_1 t)$ eventually.

1. Introduction.

We study the interacting particle system described in the first paragraph of the abstract. A construction of such a process will be discussed in the beginning of the next section.

In addition to the possible interpretations of such systems mentioned in the abstract, the B-particles have been interpreted as "packets of energy" which together with A-particles produce more energy, according to the reaction $B + A \to 2B$ (see [RS]). If memory serves us well, the study of these systems was suggested by Frank Spitzer to the first author around 1980. At that time only the case when the A and B-particles perform the same random walks (i.e., $D_A = D_B$) seems to have

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been considered. Recently, the so-called frog model – which has $D_A = 0$, i.e., the A-particles do not move – has been treated by [AMP] and [RS]. In this special case, in which the A-particles stand still, the model has subadditivity properties which were used to prove a full shape theorem. More specifically, it is proven in these references that there exists a nonrandom set B_0 such that almost surely (abbreviated to a.s. in the sequel) for all $\varepsilon > 0$

$$(1-\varepsilon)B_0 \subset \frac{1}{t}B(t) \subset (1+\varepsilon)B_0 \text{ eventually.} \tag{1.1}$$

In this paper we mainly deal with the case $D_A = D_B$. However, the upper bound for $B(t)$ (see Theorem 1 below) is relatively easy and is proven even for $D_A \neq D_B$. Probably this bound was known to several people already. It turns out that a lower bound for $B(t)$ in Theorem 2, in the case $D_A = D_B$, can be obtained by the methods of [KS]. It is still an open problem whether $B(t)$ grows linearly with t when $D_A > 0$, but $D_A \neq D_B$. In this case we can only prove that $B(t) \supset \mathcal{C}\big(K_1 t/(\log t)^p\big)$ eventually, for some constants $K_1, p > 0$. (We do not give the proof here.) With an eye towards a possible future extension to the case with $D_A \neq D_B$, some proofs here are given in greater generality than is needed for this paper.

Throughout we shall use $N_A(x,t)$ ($N_B(x,t)$) to denote the number of A-particles (respectively, B-particles) at position x at time t. N_B denotes the total number of B-particles at time 0. We always take $0 < N_B < \infty$ and consider N_B, as well as the positions of the initial B-particles, as non-random. At a site x with a B-particle at time 0 all particles immediately turn to B-particles. We write $N_A(x, 0-)$ for the number of A-particles "just before" the B-particles are added to the system. In accordance with these rules we take $N_A(x,0) = 0,\ N_B(x,0) = N_A(x,0-) + N_B(x,0)$ at a site x to which a B-particle is added at time 0. If x does not have a B-particle at time 0, then $N_A(x,0) = N_A(x,0-)$ and $N_B(x,0) = 0$. We further define

$$\widetilde{B}(t) = \{x \in \mathbb{Z}^d : \text{ a } B\text{-particle visits } x \text{ during } [0,t]\},$$

$$B(t) = \widetilde{B}(t) + \big[-\frac{1}{2}, \frac{1}{2}\big]^d,$$

and the cubes

$$\mathcal{C}(r) = [-r, r]^d. \tag{1.2}$$

Our first theorem states that the rumor/infection cannot spread from the origin faster than linearly in time.

Theorem 1. *For some constant $C_1 < \infty$, and all sufficiently large t,*

$$E\{\textit{number of } B\textit{-particles with a position outside } \mathcal{C}(C_1 t) \textit{ at time } t\} \leq 2N_B e^{-t}. \tag{1.3}$$

Consequently it is a.s. the case that

$$B(t) \subset \mathcal{C}(2C_1 t) \textit{ eventually.} \tag{1.4}$$

This result holds for any $D_A, D_B \geq 0$ and probably is even valid if one allows the A and B-particles to perform any random walk with bounded jumps of mean zero. The next theorem shows that the rumor/infection spreads at least linearly in time, but only if both the A and B-particles perform simple random walks with the same jumprate.

Theorem 2. *If $D_A = D_B$, then there exists a constant $C_2 > 0$ such that for each constant $K > 0$*

$$P\{\mathcal{C}(C_2 t) \not\subset B(t)\} \leq \frac{1}{t^K} \text{ for all large } t. \tag{1.5}$$

Consequently, a.s.

$$\mathcal{C}(C_2 t) \subset B(t) \text{ eventually.} \tag{1.6}$$

For proving a shape theorem we will need a form of Theorem 2 which also gives some information about the possible occurrence of A-particles amid the spreading B-particles. More specifically, the same proof as for Theorem 2 can be used to prove the next proposition. This answers a question raised after a lecture on this material; unfortunately we do not remember who the questioner was.

Proposition 3. *If $D_A = D_B$, then for all K there exists a constant $C_3 = C_3(K)$ such that*

$$\begin{aligned} &P\{\text{there is a vertex } z \text{ and an } A\text{-particle at the space-time point } (z,t) \text{ while} \\ &\quad \text{there also was a } B\text{-particle at } z \text{ at some time } \leq t - C_3[t\log t]^{1/2}\} \\ &\leq \frac{1}{t^K} \text{ for all sufficiently large } t. \end{aligned} \tag{1.7}$$

Consequently, for large t,

$$\begin{aligned} P\{&\text{at time } t \text{ there is a site in } \mathcal{C}(C_2 t/2) \text{ which} \\ &\qquad\qquad \text{is occupied by an } A\text{-particle}\} \leq \frac{2}{t^K}. \end{aligned} \tag{1.8}$$

Remark 1. It can be checked that the constants C_1, C_2 do not depend on the number or positions of the initial B-particles. However, the lower bounds for the times for which (1.3)-(1.6) are valid do depend on these initial data.

In Section 2 we show how our process can be constructed as a strong Markov process on a suitable state space. Much of this construction will also be used in the proof of Theorem 1 in Section 3. Section 4 contains the somewhat involved proof of Theorem 2, which relies heavily on the method of [KS] and a number of large deviation estimates for martingales. Finally, the proof of Proposition 3 is given in Section 5. Some of the detailed proofs are preceded by an outline or heuristics for the proof.

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2. Construction of a strong Markov process. Throughout this paper we make the following convention about constants. K_i will denote a strictly positive, finite constant, whose precise value is unimportant for our purposes. The value of the same K_i may be different in different formulas. We use C_i for constants whose value remains fixed throughout the paper. They will again have values in $(0, \infty)$. If necessary, we indicate on what other quantities a constant depends at the time when it is first introduced. Throughout $\|x\|$ denotes the ℓ^∞ norm of the vector $x = (x(1), \dots, x(d)) \in \mathbb{R}^d$, i.e.,

$$\|x\| = \max_{1 \le i \le d} |x(i)|. \tag{2.1}$$

$\mathbf{0}$ will denote the origin (in $\mathbb{Z}^d$ or $\mathbb{R}^d$).

In this section we shall show how to construct our process on a suitable probability space as a strong Markov process. Most readers will want to skip the proofs of Lemmas 1-3 and Proposition 4. However, the notation introduced before Lemma 1 will be used at several other places. Also, a good part of the arguments used in the construction is used again in the proof of Theorem 1.

We want to construct our process as a Markov process with a given initial state which contains only finitely many B-particles. Our first task is to choose the state space Σ_0 for our process. We shall assume that there are countably many particles in our system, which are labeled $\rho_1, \rho_2, \dots$. A particle keeps the same label throughout. The state of our system is described by specifying the location and type of each particle. We shall also add an artificial cemetery point ∂ for each particle to its coordinate space. Thus, the state space will be taken as a subset of $\Sigma := \prod_{k \ge 1} \big((\mathbb{Z}^d \cup \partial_k) \times \{A, B\}\big)$. If $\sigma = \big(\sigma'(k), \sigma''(k)\big)$ is a generic point of Σ, then $\sigma'(k)$ represents the position of ρ_k and $\sigma''(k)$ represents the type of ρ_k. Occasionally it will be more convenient to use the notation $\sigma'(\rho), \sigma''(\rho), \partial(\rho)$ for the position, type and cemetery point of a particle ρ, without specifying which of the particles ρ_k equals ρ. To describe the state space Σ_0 we introduce a process $\{Y_t\}_{t \ge 0}$. A priori, each Y_t takes values in Σ. Later we add conditions to make sure that Y_t takes values in Σ_0. We need some definitions. $\{S_t^\eta\}_{t \ge 0}$ will be a random walk with the same distribution as the random walks performed by the particles of type η (with $S_0^\eta = \mathbf{0}$, $\eta = A$ or B). We further attach to each particle ρ present at time 0 two random walk paths $t \mapsto \pi_A(t, \rho)$ and $t \mapsto \pi_B(t, \rho)$. Each $\{\pi_\eta(t, \rho)\}_{t \ge 0}$ has the same distribution as $\{S_t^\eta\}_{t \ge 0}$ and all these paths are chosen independently.

Note that we take all these paths right continuous. We write $\pi(t,\rho)$ and $\eta(t,\rho)$ for the position and type of ρ at time t, respectively.

We want to let an A-particle ρ which starts at z move along the path $t \mapsto z+\pi_A(t,\rho)$ until the time $\theta(\rho)$, say, at which it changes to a B-particle, after which it follows the path $t \mapsto z+\pi_A(\theta(\rho),\rho)+\pi_B(t,\rho)-\pi_B(\theta(\rho),\rho)$. For all ρ which have type B at time 0, we take $\theta(\rho)=0$ and let ρ move along the path $t \mapsto z+\pi_B(t,\rho)$ for all $t \geq 0$ (z again denotes the initial position of ρ). Formally, we proceed as follows. We assume that initially there are in total only finitely many B-particles, and that none of these sits at a cemetery point. We set $\tau_0=0$. Now let $k=0$, or let $k \geq 1$ and assume that we have already found the first k times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_k$ at which a B-particle has coincided with an A-particle. We also assume that at each of these times only finitely many A-particles turned into B-particles, so that at time τ_k there are still only finitely many B-particles in the system. Assume further that we have determined the paths of all particles during the interval $[0,\tau_k]$. Then we know at time τ_k which particles are B-particles and also the positions of all particles. We then assign to each particle ρ the tentative continuation of its path on $[\tau_k,\infty)$, which it would follow if it never changed type anymore. The tentative continuation of the particle paths is given by

$$\widetilde{\pi}_k(\tau_k+t,\rho)=\begin{cases} \pi(\tau_k,\rho)+[\pi_A(\tau_k+t,\rho)-\pi_A(\tau_k,\rho)] & \text{if } \eta(\tau_k,\rho)=A \\ \pi(\tau_k,\rho)+[\pi_B(\tau_k+t,\rho)-\pi_B(\tau_k,\rho)] & \text{if } \eta(\tau_k,\rho)=B. \end{cases} \tag{2.2}$$

We have to allow that some particles sit at their cemetery point. We therefore interpret the right hand side of (2.2) as $\partial(\rho)$ if $\pi(\tau_k,\rho)=\partial(\rho)$. As the reader can check in the definitions below, this has the effect that any particle stays at its cemetery point once it reaches this cemetery point. After that such a particle no longer interacts with the other particles and they play no further role in the construction of the paths of the other particles. We now use these $\widetilde{\pi}_k$ to define

$$\begin{aligned}\tau_{k+1} &= \inf\{t>\tau_k: \text{ a } B\text{-particle coincides with an } A \text{ particle at time } t \\ &\qquad\qquad \text{if the particles move according to the } \widetilde{\pi}_k\} \\ &= \inf\{t>\tau_k: \widetilde{\pi}_k(t,\rho')=\widetilde{\pi}_k(t,\rho'') \text{ for some } \rho',\rho'' \text{ with } \eta(\tau_k,\rho')=B, \\ &\qquad\qquad \eta(\tau_k,\rho'')=A\}. \end{aligned} \tag{2.3}$$

We then take

$$\pi(s,\rho)=\begin{cases} \widetilde{\pi}_k(s,\rho) & \text{for } \tau_k \leq s \leq \tau_{k+1} \text{ if } \eta(\tau_k,\rho)=A \\ \widetilde{\pi}_k(s,\rho) & \text{for } s \geq \tau_k \text{ if } \eta(\tau_k,\rho)=B. \end{cases} \tag{2.4}$$

Moreover,

$$\eta(s,\rho)=\begin{cases} A & \text{for } \tau_k \leq s < \tau_{k+1} \text{ if } \eta(\tau_k,\rho)=A \\ B & \text{for } s \geq \tau_k \text{ if } \eta(\tau_k,\rho)=B. \end{cases} \tag{2.5}$$

In addition we take $\eta(\tau_{k+1},\rho)=B$ for those ρ which have $\eta(\tau_k,\rho)=A$ and which coincide at time τ_{k+1} with a ρ' which has $\eta(\tau_k,\rho')=B$. For this special set of particles ρ we take $\theta(\rho) = \tau_{k+1}$ and call $\theta(\rho)$ the *switching time* of ρ. For all other particles their type remains unchanged at τ_{k+1}. If ρ is already of type B at time 0, then we define its switching time to be 0. Note that if there are sites with both A and B-particles in σ, then $\tau_1 = 0$ according to (2.3), and all A-particles which are at the same location as a B-particle in σ immediately change their type to B. These particles have switching time equal to 0.

These definitions give us Y_t through time τ_{k+1} and we can repeat the procedure to go till time τ_{k+2}, etc. We stop the process at

$$\widehat{\tau} := \inf\{\tau_k : \text{infinitely many } A\text{-particles turn into a } B\text{-particle at time } \tau_k \\ \text{or } \tau_{k+1} = \tau_k\}.$$

Note that a.s. $\tau_{k+1} = \tau_k$ can occur only if there are coincidences of B and A-particles immediately after τ_k, so that there must be infinitely many B-particles at $\tau_k + \varepsilon$ for any $\varepsilon > 0$. (For instance, such a situation would arise if at some time there are infinitely many particles at a site x and a B-particle adjacent to x.) We shall actually choose Σ_0 such that this possibility has probability 0. We also can not continue beyond $\tau_\infty := \lim_{k\to\infty} \tau_k$. We define for $t < \min\{\widehat{\tau}, \tau_\infty\}$,

$$\nu(t) = \text{total number of } B\text{-particles at time } t$$

and

$$Y_t'(\rho) = \pi(t, \rho),\ Y_t''(\rho) = \eta(t, \rho).$$

If $\min\{\widehat{\tau}, \tau_\infty\} > 0$ and $t \geq \min\{\widehat{\tau}, \tau_\infty\}$, then we take

$$Y_t(\rho) = \begin{cases} (\partial(\rho), A) & \text{if } \eta(s,\rho) = A \text{ for} \\ & \quad \text{all } s < \min\{\widehat{\tau}, \tau_\infty\} \\ \big(\pi(\theta(\rho), \rho) + \pi_B(t, \rho) - \pi_B(\theta(\rho), \rho), B\big) & \text{if } \theta(\rho) < \min\{\widehat{\tau}, \tau_\infty\}. \end{cases} \tag{2.6}$$

If $\min\{\widehat{\tau}, \tau_\infty\} = 0$, then we take for $t \geq 0$

$$Y_t(\rho) = \begin{cases} (\partial(\rho), A) & \text{if } \eta(0,\rho) = A \\ \big(\pi(0, \rho) + \pi_B(t, \rho), B\big) & \text{if } \eta(0,\rho) = B. \end{cases} \tag{2.7}$$

We further take

$$\nu(t) = \infty \text{ for } t \geq \min\{\widehat{\tau}, \tau_\infty\}.$$

Thus, at $\min\{\widehat{\tau}, \tau_\infty\}$ all particles which still have type A are moved to their cemetery, while the B-particles continue as B-particles along the appropriate path prescribed by their π_B. Since we start off with no B-particles at any cemetery point, the relations (2.2), (2.6) and (2.7) guarantee that there never are B-particles at the cemetery points. Thus $\nu(t)$ is actually the number of B-particles in $\mathbb{Z}^d$ at time t.

We easily see that our definitions give us the following three properties which agree with the intuitive description of our system:

$$\text{if } \rho \text{ is already of type } B \text{ at time } \tau_k, \text{ then it will stay of type } B \text{ for} \\ \text{all } t \geq \tau_k; \tag{2.8}$$

(note that if $\eta(\tau_k, \rho) = B$, then we have two possible prescriptions for $\pi(s, \rho)$ and $\eta(s, \rho)$ on $[\tau_{k+1}, \infty)$, one using (2.4) and (2.5) as written, and the other using (2.4) and (2.5) with k replaced by $k + 1$, but these two prescriptions agree)

$$\text{if } \rho \text{ has type } A \text{ at time } \tau_k, \text{ then it must have been of type } A \text{ during the whole interval } [0, \tau_k] \text{ and } \pi(s, \rho) = \pi(0, \rho) + \pi_A(s, \rho) \text{ for } s \in [0, \tau_k]; \tag{2.9}$$

$$\text{once } \rho \text{ has become of type } B\text{, then its positions change according to } \pi_B(\cdot, \rho),\ \text{i.e., } \pi(s'', \rho) - \pi(s', \rho) = \pi_B\big(s'', \rho\big) - \pi_B\big(s', \rho\big) \text{ for } s'' \geq s' \geq \theta(\rho). \tag{2.10}$$

We also point out that $\nu(t) < \infty$ for all $t < \min\{\widehat{\tau}, \tau_\infty\}$, directly from the definitions. Finally, we define

$$\Sigma_0 = \Big\{\sigma \in \big(\mathbb{Z}^d \times \{A, B\}\big)^{\mathbb{Z}_+} : 1 \leq \text{ (number of } B\text{-particles in } \sigma) < \infty, \text{ and } P\{\min\{\widehat{\tau}, \tau_\infty\} = \infty | Y_0 = \sigma\} = 1\Big\}. \tag{2.11}$$

Note that $\sigma \in \Sigma_0$ requires that none of the particles in σ are at their cemetery point.

We show in the next three lemmas and Proposition 4 that Σ_0 is a good state space for the process $\{Y_t\}$ and that $\{Y_t\}$ restricted to Σ_0 has the strong Markov property. Proposition 5 shows that under a product measure of mean μ_A Poisson variables for the numbers of A-particles on the sites of $\mathbb{Z}^d$, almost all choices lead to an initial point in Σ_0. In particular $\Sigma_0 \neq \emptyset$. Note, however, that in Lemmas 1 - 3 and Proposition 4 the numbers of initial A-particles at the various sites are not random. The initial state there is any point of Σ or Σ_0, respectively. The basic σ-fields which we shall use are

$$\mathcal{F}_t^0 := \sigma\text{-field generated by } \{Y_s : s \leq t\}. \tag{2.12}$$

The elements of these σ-fields are subsets of $\Sigma^{[0,\infty)}$, the path space for $\{Y_t\}_{t\geq 0}$. The coordinate spaces of Σ, i.e., the spaces $(\mathbb{Z}^d \cup \partial_k) \times \{A, B\}$, are countable. We endow them with the discrete topology and use the product of these topologies on Σ.

Unfortunately the description of Σ_0 is not very explicit, and it may seem useless to go through such length to find such a state space. Instead one might choose to work only with the process starting with independent Poisson numbers of particles at the sites of $\mathbb{Z}^d$. However, we know of no way to prove that such a process has the strong Markov property without describing the state space Σ_0, and our proofs use the strong Markov property at several places.

Lemma 1. *The process $\{Y_t\}_{t\ge 0}$ is a Markov process on Σ with respect to the filtration $\{\mathcal{F}_t^0\}_{t\ge 0}$. Its transition function equals*

$$Q_s(\sigma,\Gamma) = P\{Y_s \in \Gamma | Y_0 = \sigma\},\ s\ge 0,\ \Gamma\subset\Sigma. \tag{2.13}$$

Moreover, $t\mapsto Y_t$ is right continuous if we use the product topology on Σ.

Proof. For all but the last continuity statement, it suffices to prove that for given $0 < t_1 < t_2 < \cdots < t_k$ and $\Gamma_i \subset \Sigma$,

$$\begin{aligned} P\{Y_{t_i} \in \Gamma_i, 1\le i\le k | Y_0 = \sigma\} \\ = \int_{\sigma_1\in\Gamma_1} \cdots \int_{\sigma_k\in\Gamma_k} Q_{t_1}(\sigma, d\sigma_1)\cdots Q_{t_k - t_{k-1}}(\sigma_{k-1}, d\sigma_k). \end{aligned} \tag{2.14}$$

It follows from the definition (2.2) that for given t_{i-1}, t_i and $Y_{t_{i-1}}$, the state Y_{t_i} depends only on $Y_{t_{i-1}}$ and the increments $\pi_\eta(s,\rho) - \pi_\eta(t_{i-1},\rho)$ for $\eta = A$ or B, $t_{i-1} < s \le t_i$ and ρ any particle. Write ζ_i for the collection of paths $s\mapsto \pi_\eta(s,\rho_k) - \pi_\eta(t_{i-1},\rho_k)$, $t_{i-1} < s \le t_i, k\ge 1$. Thus ζ_i takes values in the space $\big[\text{all paths from } (t_{i-1}, t_i] \text{ to } \mathbb{Z}^d \times \{A,B\}\big]^{\mathbb{Z}_+}$. There exist some sets $\Delta_i = \Delta_i(Y_{t_{i-1}})$ in these spaces, such that (still for the given t_{i-1}, t_i and $Y_{t_{i-1}}$) $Y_{t_i}\in\Gamma_i$ if and only if $\zeta_i \in \Delta_i$. Consequently,

$$P\{Y_{t_i}\in\Gamma_i,\ 1\le i\le k | Y_0 = \sigma\} = P\{\zeta_i \in \Delta_i(Y_{t_{i-1}}),\ 1\le i\le k|Y_0 = \sigma\}.$$

But the ζ_i for different i are independent, and by definition of Q, $P\{\zeta_i\in\Delta_i(y)\} = Q_{t_i - t_{i-1}}(y,\Gamma_i)$. (2.14) now follows by induction on k and standard measure theory (see for instance Theorem 20.3 in [B]).

Lastly, the right continuity of Y follows from the right continuity of each $\pi_\eta(\cdot,\rho)$ and the fact that $Y'(\rho)$ consists of appropriate pieces of $\pi_A(\cdot,\rho)$, $\pi_B(\cdot,\rho)$ and the constant $\partial(\rho)$ on the intervals $\big[0, \min\{\theta(\rho), \widehat{\tau}, \tau_\infty\}\big)$, $\big[\theta(\rho), \min\{\widehat{\tau}, \tau_\infty\}\big)$ and $\big[\min\{\widehat{\tau},\tau_\infty\},\infty\big)$. Also $Y_t''(\rho) = \eta(\rho)$ is constant on these left closed, right open intervals. ∎

To formulate the next lemma we define

$$\alpha_t(z) = P\{S_t^A = -z\}, \tag{2.15}$$

and

$$M_s(\sigma) = \sum_{z\in\mathbb{Z}^d} \alpha_s(z) \sum_{\rho:\sigma(\rho)=(z,A)} 1 = \sum_\rho I[\sigma''(\rho) = A]\alpha_s(\sigma'(\rho)). \tag{2.16}$$

For purposes of comparison it is useful to couple our system with the system in which there are no B-particles and in which all original A-particles move forever without interaction. In this system, which we shall denote by $\mathcal{P}^*$, an A-particle ρ which starts at z will have position $z + \pi_A(t,\rho)$ for all t. Thus it coincides with this same particle in the Y-process until $\theta(\rho)$. After this time, the increments of ρ in the Y-process will be the same as those of $\pi_B(\cdot, \rho)$, but in the $\mathcal{P}^*$ system, the increments of ρ will be the same as of $\pi_A(\cdot, \rho)$. We write $N^*(x, t)$ for the number of particles at the space-time point (x, t) in the system $\mathcal{P}^*$. $N^*(x, 0)$ is taken equal to $N_A(x, 0-)$, the initial number of A-particles at x, but the initial B-particles are immediately removed in $\mathcal{P}^*$ and never change the type of an A-particles in $\mathcal{P}^*$. $N^*(x, t)$ is an upper bound for the number of A-particles at (x, t) in our original system, because in that system A-particles can turn into B-particles at some time, after which they are no longer counted in N_A. Thus

$$\sum_{\rho: Y_t(\rho) = (x, A)} 1 \le N^*(x, t). \tag{2.17}$$

One more piece of notation. We shall write P^σ for the measure governing the process $\{Y_t\}$ given that it starts with $Y_0 = \sigma$. This is the unique measure on the space of right continuous paths into Σ with finite dimensional distributions given by (2.13) and (2.14). E^σ denotes expectation with respect to P^σ.

Lemma 2. *Fix the initial state $\sigma \in \Sigma_0$. Then almost surely $[P^\sigma]$ the following properties hold:*

$$\min\{\widehat{\tau}, \tau_\infty\} = \infty; \tag{2.18}$$

$$M_s(Y_t) < \infty \text{ for all } s, t \ge 0; \tag{2.19}$$

and

$$\sup_{s \le t} \big(\text{number of particles at } (z, s)\big) < \infty \text{ for all } z \in \mathbb{Z}^d, t \ge 0. \tag{2.20}$$

Proof. We fix $\sigma \in \Sigma_0$. All probabilities in this proof will be conditioned on $\{Y_0 = \sigma\}$.

First note that property (2.18) is a matter of definition only, since $\sigma \in \Sigma_0$ requires that

$$P^\sigma\big\{\min\{\widehat{\tau}, \tau_\infty\} = \infty\big\} = 1.$$

An important consequence of (2.18) is that, almost surely, there are at all times a nonzero, but finite, number of B-particles in the system.

To start the proof of (2.19) we show that

$$M_s(\sigma) < \infty \text{ for all } 0 \le s < \infty \tag{2.21}$$

(note that $M_s(\sigma)$ is nonrandom when σ is fixed). Assume that this does not hold and $M_s(\sigma) = \infty$ for some s. Since there is almost surely at least one B-particle present at time $s+1$, there exists an $x \in \mathbb{Z}^d$ such that $P^\sigma\{$there is a B-particle at $(x, s+1)\} > 0$. Note next that α has the following simple properties:

$$\alpha_{t+u}(z) \ge e^{-D_A u}\alpha_t(z), \tag{2.22}$$

and consequently $M_{t+u}(\sigma) \geq e^{-D_A u} M_t(\sigma)$. Also

$$\alpha_{t+s}(z) \geq \alpha_t(z')\alpha_s(z-z'). \tag{2.23}$$

Therefore $M_s(\sigma) = \infty$ implies

$$\sum_{z\in\mathbb{Z}^d} \sum_{\rho:\sigma(\rho)=(z,A)} \alpha_{s+1}(z-x)$$
$$\geq \alpha_1(-x) \sum_{z\in\mathbb{Z}^d} \sum_{\rho:\sigma(\rho)=(z,A)} \alpha_s(z) = \alpha_1(-x) M_s(\sigma) = \infty.$$

The left hand side equals the expected number of initial A-particles $\rho \in \mathbb{Z}^d$ that satisfy $z(\rho) + \pi_A(s+1,\rho) = x$, where $z(\rho)$ is the initial position of ρ. The events $z(\rho)+\pi_A(s+1,\rho) = x$ for different ρ are independent, so that by the Borel-Cantelli lemma we have that almost surely infinitely many of these events occur. But, on the event {there is a B-particle at $(x, s+1)$}, any ρ for which $z(\rho)+\pi_A(s+1,\rho) = x$ occurs meets a B-particle at or before time $s+1$, so that infinitely many B-particles arise during $[0, s+1]$. This contradicts (2.18) because, as we already pointed out, $\nu(u) < \infty$ for $u < \min\{\widehat{\tau}, \tau_\infty\}$.

Now assume, to arrive at a contradiction, that for some $s \geq 0$, $P^\sigma\{M_s(Y_t) = \infty$ for some $t \geq 0\} = C > 0$. Then there exist some $D < \infty$ and for each $E < \infty$ an $F = F(D,E)$ such that

$$P^\sigma\Big\{ \sum_{z\in\mathcal{C}(F)} \sum_\rho I[Y'_t(\rho) = z] I[Y''_t(z) = A]\alpha_s(z) \geq E+1 \text{ for some } t \leq D-1\Big\} \geq \frac{C}{2}.$$

(Here and in the rest of the paper we abbreviate $\sum_{z\in\mathcal{A}\cap\mathbb{Z}^d}$ to $\sum_{z\in\mathcal{A}}$ in cases where confusion is unlikely. Similarly we abbreviate $\cup_{z\in\mathcal{A}\cap\mathbb{Z}^d}$ to $\cup_{z\in\mathcal{A}}$.) Since the paths π_η are all right continuous, and $\sum_\rho I[Y'_u(\rho) = z] I[Y''_u(z) = A] \leq N^*(z,u)$, we even have for some n

$$P^\sigma\Big\{ \sum_{z\in\mathcal{C}(F)} N^*(z, k2^{-n})\alpha_s(z) \geq E \text{ for some } k \leq D2^n\Big\} \geq \frac{C}{4}. \tag{2.24}$$

It is not hard to see that the process $\{N^*(z,t) : z \in \mathbb{Z}^d\}_{t\geq 0}$ has the (simple) Markov property with respect to the σ-fields $\mathcal{F}^*_t :=$ σ-field generated by $\{N^*(y,s) : y \in \mathbb{Z}^d,\ s \leq t\}$ (see also the proof of Lemma 1). In particular, if we define

$$k_0 = \inf\Big\{k : \sum_{z\in\mathcal{C}(F)} N^*(z, k2^{-n})\alpha_s(z) \geq E\Big\},$$

then for $k \leq D2^n$, on the event $\{k_0 = k\}$,

$$E^\sigma\big\{N^*(\mathbf{0}, D+s) | \mathcal{F}^*_{k2^{-n}}\big\} = \sum_{z\in\mathbb{Z}^d} N^*(z, k2^{-n})\alpha_{D+s-k2^{-n}}(z)$$
$$\geq e^{-DD_A} \sum_{z\in\mathbb{Z}^d} N^*(z, k2^{-n})\alpha_s(z) \geq e^{-DD_A} E.$$

This would imply $E^\sigma\{N^*(\mathbf{0}, D+s)\} \ge (CE/4)\exp[-DD_A]$. Since E was arbitrary, this would even imply

$$E^\sigma\{N^*(\mathbf{0}, D+s)\} = \infty$$

if $C > 0$. But the left hand side here equals

$$\sum_{z\in\mathbb{Z}^d} E^\sigma\{N^*(z,0)\}\alpha_{D+s}(z) = \sum_{z\in\mathbb{Z}^d} I[\sigma(\rho) = (z,A)]\alpha_{D+s}(z) = M_{D+s}(\sigma).$$

Thus we finally arrived at a contradiction to (2.21) from the assumption $C > 0$. This proves that for each fixed s, a.s. $[P^\sigma]$, $M_s(Y_t) < \infty$, simultaneously for all t. This then also holds simultaneously for all integer $s \ge 0$, and by virtue of (2.22) even simultaneously for all s. Thus (2.19) holds.

Next we turn to (2.20). We already know (2.18) and that $\nu(t) < \infty$ for $t < \min\{\widehat{\tau}, \tau_\infty\}$. Since $\nu(t)$ is nondecreasing in t this shows that $\sup_{s\le t}\nu(s) < \infty$ for all $t < \infty$. Thus we only have to count A-particles in (2.20). But the number of A particles at (z,s) is bounded by $N^*(z,s)$. It therefore suffices to show that for each fixed (z,t)

$$\sup_{s\le t} N^*(z,s) < \infty \text{ a.s.} \tag{2.25}$$

To see that this is the case we note that

$$\begin{aligned} E^\sigma\Big\{\sup_{s\le t} N^*(z,s)\Big\} &\le E^\sigma\{(\text{number of } A\text{-particles in } \mathcal{P}^* \text{ which visit } z \text{ during } [0,t])\} \\ &\le \sum_{y\in\mathbb{Z}^d}\sum_{\rho:\sigma(\rho)=(y,A)} P\{y+\pi_A(s,\rho) = z \text{ for some } s\le t\}. \end{aligned} \tag{2.26}$$

But, if ρ starts at y with type A, then

$$\begin{aligned} \int_0^{t+1}\alpha_s(y-z)ds &= E\{\text{amount of time spent by } \rho \text{ at } z \text{ during } [0,t+1] \text{ in } \mathcal{P}^*\} \\ &\ge P\{y+\pi_A(s,\rho) \text{ reaches } z \text{ at some } s\le t \text{ and stays at } z \\ &\qquad\qquad \text{for at least one unit of time}\} \\ &\ge e^{-D_A}P\{y+\pi_A(s,\rho) = z \text{ for some } s\le t\}. \end{aligned} \tag{2.27}$$

Thus (2.26) combined with (2.22) and (2.23) shows that

$$\begin{aligned} &E^\sigma\Big\{\sup_{s\le t} N^*(z,s)\Big\} \\ &\le e^{D_A}\sum_{y\in\mathbb{Z}^d}\sum_\rho I[\sigma(\rho)=(y,A)]\int_0^{t+1}\alpha_s(y-z)ds \\ &\le e^{D_A}\int_0^{t+1} e^{(t+1)D_A}\sum_{y\in\mathbb{Z}^d}\sum_\rho I[\sigma(\rho)=(y,A)]\alpha_{t+1}(y-z)ds \\ &\le e^{(t+2)D_A}[\alpha_1(z)]^{-1}\int_0^{t+1}\sum_{y\in\mathbb{Z}^d}\sum_\rho I[\sigma(\rho)=(y,A)]\alpha_{t+2}(y)ds \\ &= (t+1)e^{(t+2)D_A}[\alpha_1(z)]^{-1}M_{t+2}(\sigma) < \infty. \end{aligned} \tag{2.28}$$

Of course (2.25), and (2.20), follow from this. ∎

In the next proposition we shall prove that the Y-process stays in Σ_0 if it starts at a $\sigma \in \Sigma_0$, and that the process on Σ_0 has the strong Markov property. The next lemma proves a kind of continuity property which is useful in proving these properties. For $L, K, T \ge 0$ we define

$$N_B\big(\mathcal{C}^c(L), s\big) = \text{number of } B \text{ particles outside } \mathcal{C}(L) \text{ at time } s,$$

$$\begin{aligned}\mathcal{H}(1, L, T) &= \big\{ \sup_{s<T} N_B(\mathcal{C}^c(L), s) \ge 1 \big\} \\ &= \{\text{there are some } B \text{ particles outside } \mathcal{C}(L) \text{ during } [0, T)\}.\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\mathcal{H}(2, L, K, T) = \big\{&\text{some } A\text{-particle which starts outside } \mathcal{C}(K) \text{ enters } \mathcal{C}(L) \\ &\text{during } [0, T], \text{ still as an } A\text{-particle}\big\}.\end{aligned} \tag{2.30}$$

Lemma 3. *Let $\sigma_1, \sigma_2 \in \Sigma$ and $T < \infty, 0 \le L \le K$ satisfy*

$$\sigma_i'(\rho) \in \mathbb{Z}^d \text{ for all } \rho,\ i = 1, 2, \tag{2.31}$$

there are only finitely many B-particles in σ_i, and these lie in $\mathcal{C}(L)$, $i = 1, 2$, (2.32)

and

$$\sigma_1(\rho) = \sigma_2(\rho) \text{ for all } \rho \in \mathcal{C}(K), \tag{2.33}$$

i.e., if $\sigma_i'(\rho) \in \mathcal{C}(K)$ for $i = 1$ or 2, then $\sigma_1(\rho) = \sigma_2(\rho)$. Finally, let $\mathcal{E}$ be an event which is a finite union of events of the form

$$\{Y_{s_j}(\rho_j) = (z_j, \eta_j), 1 \le j \le n\} = \{\pi(s_j, \rho_j) = z_j, \eta(s_j, \rho_j) = \eta_j, 1 \le j \le n\} \tag{2.34}$$

for some fixed $z_j \in \mathbb{Z}^d, \eta_j \in \{A, B\}$, $s_j < T$, and such that in σ_1 and σ_2

$$\rho_j \in \mathcal{C}(K) \text{ for } 1 \le j \le n. \tag{2.35}$$

Then

$$\begin{aligned}&\big|P^{\sigma_1}\{\mathcal{E}\} - P^{\sigma_2}\{\mathcal{E}\}\big| \\ &\quad \le P^{\sigma_2}\{\mathcal{H}(1, L, T)\} + P^{\sigma_1}\{\mathcal{H}(2, L, K, T)\} + 2P^{\sigma_2}\{\mathcal{H}(2, L, K, T)\}.\end{aligned} \tag{2.36}$$

Proof. To prove this lemma we couple two processes $\{Y_t^i\}$, $i = 1, 2$, starting from σ_1 and σ_2, respectively, by using the same paths $\pi_A(\cdot, \rho), \pi_B(\cdot, \rho)$ for all $\rho \in \mathcal{C}(K)$

(these particles are necessarily the same in σ_1 and σ_2 by (2.33)), and independent copies $\pi_A^i(\cdot,\rho), \pi_B^i(\cdot,\rho),\ i=1,2$ for $\rho \notin \mathcal{C}(K)$ in σ_1 or σ_2. Let $\mathcal{H}^i(1,L,T)$ be the event that $\mathcal{H}(1,L,T)$ occurs for the process $\{Y^i\}$ in this coupling, and similarly for $\mathcal{H}^i(2,L,K,T)$ and $\mathcal{E}^i$. From the paths $\pi_A(\cdot,\rho), \pi_B(\cdot,\rho)$ for $\rho \in \mathcal{C}(K)$ we can construct paths (including types) for the particles $\rho \in \mathcal{C}(K)$ by using the rules (2.2)-(2.6), but *ignoring all particles which start outside* $\mathcal{C}(K)$ in either process $Y^i,\ i=1,2$. In other words, we want to act as if we start only with the finitely many particles in $\mathcal{C}(K)$. We denote the resulting position and type of ρ at time s by $\pi^K(s,\rho)$ and $\eta^K(s,\rho)$. Because of the coupling these do not depend on i. We further write $\widetilde{\mathcal{E}}^K$ for the event that (2.34) occurs if π,η there are replaced by π^K,η^K. Finally, let

$$\begin{aligned}\widetilde{\mathcal{H}}^K(1,L,T) = \text{there exist } s<T \text{ for which } &\pi^K(s,\rho) \notin \mathcal{C}(L),\\ &\eta^K(s,\rho)=B \text{ for some } \rho \text{ starting in } \mathcal{C}(K) \text{ when}\\ &\text{using only the paths } \big(\pi^K(\cdot,\rho),\eta^K(\cdot,\rho)\big).\end{aligned}$$

Now clearly

$$\begin{aligned}&\big|P^{\sigma_1}\{\mathcal{E}\} - P^{\sigma_2}\{\mathcal{E}\}\big| = \big|P\{\mathcal{E}^1\} - P\{\mathcal{E}^2\}\big|\\ &\le \big|P\{\mathcal{E}^1 \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\mathcal{E}^2 \setminus \widetilde{\mathcal{H}}^K(1,L,T)\}\big| + P\{\widetilde{\mathcal{H}}^K(1,L,T)\}.\end{aligned} \tag{2.37}$$

But

$$\widetilde{\mathcal{E}}^K \setminus \big(\widetilde{\mathcal{H}}^K(1,L,T) \cup \mathcal{H}^i(2,L,K,T)\big) \subset \mathcal{E}^i \setminus \widetilde{\mathcal{H}}^K(1,L,T), \tag{2.38}$$

because on the complement of $\widetilde{\mathcal{H}}^K(1,L,T) \cup \mathcal{H}^i(2,L,K,T)$ the A-particles which start outside $\mathcal{C}(K)$ cannot interact with any B-particles during $[0,T)$. Indeed, the latter particles are formed by means of the paths $\big(\pi^K(\cdot,\rho),\eta^K(\cdot,\rho)\big)$ and only appear in $\mathcal{C}(L)$ during $[0,T)$, while the A-particles which start outside $\mathcal{C}(K)$ do not enter $\mathcal{C}(L)$ and remain of type A during $[0,T]$. Thus, the type of the particles which start in $\mathcal{C}(K)$ is not influenced by the particles which start outside $\mathcal{C}(K)$. This gives (2.38) and consequently

$$P\{\mathcal{E}^i \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} \ge P\{\widetilde{\mathcal{E}}^K \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\mathcal{H}^i(2,L,K,T)\}.$$

The same argument shows that

$$P\{\widetilde{\mathcal{E}}^K \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} \ge P\{\mathcal{E}^i \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\mathcal{H}^i(2,L,K,T)\}$$

and hence

$$\begin{aligned}&\big|P\{\mathcal{E}^i \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\widetilde{\mathcal{E}}^K \setminus \widetilde{\mathcal{H}}^K(1,L,T)\}\big|\\ &\qquad \le P\{\mathcal{H}^i(2,L,K,T)\} = P^{\sigma_i}\{\mathcal{H}(2,K,L,T)\}.\end{aligned} \tag{2.39}$$

Still by the same argument,

$$\begin{aligned}&P\{\widetilde{\mathcal{H}}^K(1,L,T) \text{ does not occur}\}\\&\geq P\{\mathcal{H}^i(1,L,T) \text{ does not occur}\} - P\{\mathcal{H}^i(2,L,K,T)\}\\&= P^{\sigma_i}\{\mathcal{H}(1,L,T) \text{ does not occur}\} - P^{\sigma_i}\{\mathcal{H}(2,L,K,T)\},\end{aligned}$$

so that

$$P\{\widetilde{\mathcal{H}}^K(1,L,T)\} \leq P^{\sigma_2}\{\mathcal{H}(1,L,T)\} + P^{\sigma_2}\{\mathcal{H}(2,L,K,T)\}. \tag{2.40}$$

Finally,

$$\begin{aligned}&\big|P\{\mathcal{E}^1 \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\mathcal{E}^2 \setminus \widetilde{\mathcal{H}}^K(1,L,T)\}\big|\\&\leq \big|P\{\widetilde{\mathcal{E}}^K \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\widetilde{\mathcal{E}}^K \setminus \widetilde{\mathcal{H}}^K(1,L,T)\}\big|\\&\qquad + \big|P\{\mathcal{E}^1 \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\widetilde{\mathcal{E}}^K \setminus \widetilde{\mathcal{H}}^K(1,L,T)\}\big|\\&\qquad + \big|P\{\mathcal{E}^2 \setminus \widetilde{\mathcal{H}}^K(1,L,T)\} - P\{\widetilde{\mathcal{E}}^K \setminus \widetilde{\mathcal{H}}^K(1,L,T)\}\big|,\end{aligned}$$

so that (2.36) follows with the help of (2.37)-(2.40). ■

Proposition 4. *For each $\sigma \in \Sigma_0$ one has*

$$P^\sigma\{Y_t \in \Sigma_0 \text{ for all } t \geq 0\} = 1. \tag{2.41}$$

Also, a.s. $[P^\sigma]$,

$$\text{for all } t, s \geq 0, \quad P^{Y_t}\{Y_s'(\rho) = \partial(\rho) \text{ for some } \rho\} \leq P^{Y_t}\{\min\{\widehat{\tau}, \tau_\infty\} \leq s\} = 0. \tag{2.42}$$

Moreover, if $\mathcal{E}$ is a finite union of sets of the form (2.34), then

$$t \mapsto P^{Y_t}\{\mathcal{E}\} \text{ is right continuous a.s. } [P^\sigma]. \tag{2.43}$$

The process $\{Y_t\}$ starting at $\sigma \in \Sigma_0$ has the strong Markov property with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where

$$\mathcal{F}_t := \cap_{h>0}\mathcal{F}_t^0. \tag{2.44}$$

Proof. Unfortunately this proposition has a somewhat circuitous proof. We divide it into 3 steps.
Step 1. Here we prove that a.s. $[P^\sigma]$, for each fixed T_1, T_2, L,

$$\lim_{K\to\infty} \sup_{t\leq T_2} P^{Y_t}\{\mathcal{H}(2,L,K,T_1)\} = 0. \tag{2.45}$$

We prove this by showing that for all $\varepsilon_1, \varepsilon_2 > 0$, $T_1, T_2, L > 0$ there exists a $K_0 = K_0(\sigma, \varepsilon_i, T_i, L) < \infty$ such that for $K \geq K_0$

$$P^\sigma\Big\{\exists_{t\leq T_2} \quad P^{Y_t}\{\mathcal{H}(2,L,K,T_1)\} \geq \varepsilon_1\Big\} \leq \varepsilon_2. \tag{2.46}$$

To this end we observe that

$$\begin{aligned}
&P^{Y_t}\{\text{some } A\text{-particle which is outside } \mathcal{C}(K) \text{ in the state } Y_t \\
&\qquad\qquad\qquad\text{enters } \mathcal{C}(L) \text{ during } [0,T_1], \text{ still as an } A\text{-particle}\} \\
&\le E^{Y_t}\{\text{number of particles outside } \mathcal{C}(K) \text{ which} \\
&\qquad\qquad\qquad\text{enter } \mathcal{C}(L) \text{ as } A\text{-particles during } [0,T_1]\} \\
&\le \sum_{z\notin\mathcal{C}(K)} N^*(z,t) \sum_{u\in\mathcal{C}(L)} P\{\text{an } A\text{-particle in } \mathcal{P}^* \text{ which starts} \\
&\qquad\qquad\qquad\text{at } z \text{ visits } u \text{ during } [0,T_1]\} \\
&\le \sum_{z\notin\mathcal{C}(K)} N^*(z,t) \sum_{u\in\mathcal{C}(L)} e^{(T_1+2)D_A}(T_1+1)\alpha_{T_1+1}(z-u) \text{ (see (2.27) and (2.22)).}
\end{aligned}$$

The supremum over $t \le T_2$ of the right hand side here has expectation at most

$$\begin{aligned}
&\sum_{z\notin\mathcal{C}(K)} \sum_{u\in\mathcal{C}(L)} e^{(T_1+2)D_A}(T_1+1)\alpha_{T_1+1}(z-u)E^{\sigma}\{\sup_{t\le T_2} N^*(z,t)\} \\
&\le \sum_{z\notin\mathcal{C}(K)} \sum_{u\in\mathcal{C}(L)} e^{(T_1+2)D_A}(T_1+1)\alpha_{T_1+1}(z-u)(T_2+1)e^{(T_2+2)D_A} \\
&\qquad\qquad \sum_{y\in\mathbb{Z}^d}\sum_{\rho} I[\sigma(\rho)=(y,A)]\alpha_{T_2+1}(y-z) \text{ (see (2.28)).}
\end{aligned} \tag{2.47}$$

If we sum here over all z instead of only over $z \notin \mathcal{C}(K)$, then we obtain

$$\begin{aligned}
&(T_1+1)(T_2+1)e^{(T_1+T_2+4)D_A} \sum_{u\in\mathcal{C}(L)}\sum_{y\in\mathbb{Z}^d}\sum_{\rho} I[\sigma(\rho)=(y,A)] \\
&\qquad\qquad \times \sum_{z\in\mathbb{Z}^d} \alpha_{T_1+1}(z-u)\alpha_{T_2+1}(y-z)) \\
&= (T_1+1)(T_2+1)e^{(T_1+T_2+4)D_A} \sum_{u\in\mathcal{C}(L)}\sum_{y\in\mathbb{Z}^d}\sum_{\rho} I[\sigma(\rho)=(y,A)]\alpha_{T_1+T_2+2}(y-u) \\
&\le (T_1+1)(T_2+1)e^{(T_1+T_2+4)D_A} \sum_{u\in\mathcal{C}(L)} [\alpha_1(u)]^{-1} M_{T_1+T_2+3}(\sigma) \text{ (see (2.23))} \\
&< \infty.
\end{aligned}$$

Thus by the dominated convergence theorem the expression (2.47) can be made arbitrarily small by taking K large. In particular we can make (2.47) smaller than $\varepsilon_1\varepsilon_2$. (2.46) then follows from Markov's inequality, and (2.45) follows from (2.46) and the Borel-Cantelli lemma.

Step 2. Our second step is to prove that for all $\sigma \in \Sigma_0$, $\varepsilon_1, \varepsilon_2 > 0$ and $0 \le T_1, T_2 < \infty$, there exists an $L = L(\sigma, \varepsilon_i, T_i)$ such that

$$P^{\sigma}\Big\{\exists_{t\le T_2} \quad P^{Y_t}\{\mathcal{H}(1,L,T_1)\} \ge \varepsilon_1\Big\} \le \varepsilon_2. \tag{2.48}$$

We shall give a proof by contradiction, so we assume that (2.48) fails. Then we can fix $\sigma \in \Sigma_0$, $\varepsilon_i > 0, T_i < \infty$ such that for all L

$$P^\sigma\Big\{\exists_{t\le T_2} \quad P^{Y_t}\{\sup_{s<T_1} N_B(\mathcal{C}^c(L),s) \ge 1\} \ge \varepsilon_1\Big\} > \varepsilon_2. \tag{2.49}$$

The main step is to show that then for all L there also exists an $n = n(L) = n(L, T_1, T_2, \varepsilon_1, \varepsilon_2)$ for which

$$P^\sigma\Big\{\exists_{k\le T_2 2^{n+1}} \quad P^{Y_{k2^{-n}}}\{\sup_{s<T_1} N_B(\mathcal{C}^c(L),s) \ge 1\} \ge \frac{\varepsilon_1}{4}\Big\} > \frac{\varepsilon_2}{2}. \tag{2.50}$$

We deduce this from an application of Lemma 3 with take $\sigma_1 = Y_t$ and $\sigma_2 = Y_{t+h}$ for some $t \le T_2$. We shall let $h \downarrow 0$. First we check that a.s. $[P^\sigma]$ (2.31) and the first part of (2.32) are satisfied for these σ_i. Indeed, $\sigma \in \Sigma_0$, and by definition this means that we start with all particles at points of $\mathbb{Z}^d$ and not at their cemetery point. By the construction of the Y-process we then have that no particle reaches its cemetery point before $\min\{\widehat{\tau}, \tau_\infty\}$, and this time is infinite a.s. $[P^\sigma]$, by (2.18). Consequently, $Y'_t(\rho) \ne \partial(\rho)$ for all $t \ge 0$ and all ρ. Moreover, there are only finitely many B-particles at all times, since $\tau_\infty = \infty$ a.s. $[P^\sigma]$.

We note in passing that the same argument shows that a.s. $[P^\sigma]$

$$P^{Y_t}\{Y'_s(\rho) = \partial(\rho) \text{ for some } \rho\} \le P^{Y_t}\{\min\{\widehat{\tau}, \tau_\infty\} \le s\}. \tag{2.51}$$

Indeed, to apply the preceding argument we only need to know that a.s. $[P^\sigma]$, for all $t \ge 0$, in Y_t, there are no particles at their cemetery and there are only finitely many B-particles. But we just established that there are no particles at their cemetery, and there are only finitely many B-particles at a finite t, because $P^\sigma\{\tau_\infty < \infty\} = 0$. Thus (2.51) holds and this is the first inequality in (2.42).

Next, by right continuity of $s \mapsto Y_s$, we have for each fixed ρ that $Y_{t+h}(\rho) = Y_t(\rho)$ for all sufficiently small $h > 0$. Therefore, a.s. $[P^\sigma]$, if $\mathcal{E}$ is a finite union of sets of the form (2.34), then there exist some $K_1(\mathcal{E})$, and for all $K \ge K_1$ some (random) $h_1 = h_1(K,t) > 0$, such that conditions (2.33) and (2.35) are satisfied for $K \ge K_1, 0 < h \le h_1$. K_1 merely has to be such that all ρ which appear in the description of $\mathcal{E}$ belong to $\mathcal{C}(K_1)$ in the state Y_t, and $h_1 \le 1$ such that $Y_{t+h}(\rho) = Y_t(\rho)$ for all $\rho \in \mathcal{C}(K_1)$ and $h \le h_1$. Then, by (2.36), for $K \ge K_1 \vee L$ and $0 < h \le h_1$,

$$\begin{aligned} P^{Y_t}\{\mathcal{E}\} \le & P^{Y_{t+h}}\{\mathcal{E}\} + P^{Y_{t+h}}\{\mathcal{H}(1,L,T_1)\} \\ & + P^{Y_t}\{\mathcal{H}(2,L,K,T_1)\} + 2P^{Y_{t+h}}\{\mathcal{H}(2,L,K,T_1)\} \\ & + I[Y_s \text{ has some } B\text{-particles outside } \mathcal{C}(L) \text{ for some } s \le T_2 + 1]. \end{aligned} \tag{2.52}$$

The indicator function in the right hand side here has been inserted to take care of the second part of (2.32) which requires in the present application of Lemma 3 that all B-particles in Y_t and Y_{t+h} lie in $\mathcal{C}(L)$. We apply (2.52) with the following choice: fix L and take $L' > L$ and $K > L$ and

$$\mathcal{E} = \mathcal{E}(N, L', K) = \bigcup_{k \le N} \bigcup_{z \in \mathcal{C}(L') \setminus \mathcal{C}(L)} \bigcup_{\rho \in \mathcal{C}(K)} \{\pi(r_k, \rho) = z, \eta(r_k, \rho) = B\}$$

with $r_1, r_2, \dots$ some enumeration of the rationals in $[0, T_1)$. Then $\mathcal{E} \subset \mathcal{H}(1, L, T_1)$. On the other hand, for given $\varepsilon > 0$ there exist N, L', K large enough such that

$$P^{Y_t}\{\mathcal{E}\} \ge P^{Y_t}\{\mathcal{H}(1, L, T_1)\} - \varepsilon.$$

Consequently, from (2.52),

$$\begin{aligned} P^{Y_t}\{\mathcal{H}(1, L, T_1)\} - \varepsilon \le 2 \liminf_{h \downarrow 0} P^{Y_{t+h}}\{\mathcal{H}(1, L, T_1)\} \\ + P^{Y_t}\{\mathcal{H}(2, L, K, T_1)\} + 2 \limsup_{h \downarrow 0} P^{Y_{t+h}}\{\mathcal{H}(2, L, K, T_1)\} \\ + I[Y_s \text{ has some } B\text{-particles outside } \mathcal{C}(L) \text{ for some } s \le T_2 + 1]. \end{aligned}$$

We now first take $K \to \infty$ and apply (2.45). This gets rid of the second and third term in the right hand side of the inequality above. Next we let $\varepsilon \downarrow 0$ to obtain

$$\begin{aligned} \liminf_{h \downarrow 0} P^{Y_{t+h}}\{\mathcal{H}(1, L, T_1)\} \ge \frac{1}{2} P^{Y_t}\{\mathcal{H}(1, L, T_1)\} \\ - I[Y_s \text{ has some } B\text{-particles outside } \mathcal{C}(L) \text{ for some } s \le T_2 + 1] \text{ a.s. } [P^\sigma]. \end{aligned} \tag{2.53}$$

Finally we observe that the fact that a.s. $[P^\sigma]$ there are only finitely many B-particles in Y_s for $s \le T_2 + 1$ (see (2.41)) implies that

$$\lim_{L \to \infty} I[Y_s \text{ has some } B\text{-particles outside } \mathcal{C}(L) \text{ for some } s \le T_2 + 1] = 0 \tag{2.54}$$

a.s. $[P^\sigma]$. Therefore, (2.49) and (2.53) together imply that there exists an $L_1 < \infty$ such that for $L \ge L_1$

$$P^\sigma\{\exists_{t \le T_2} \liminf_{h \downarrow 0} P^{Y_{t+h}}\{\mathcal{H}(1, L, T_1)\} \ge \frac{\varepsilon_1}{2}\} \ge \frac{3\varepsilon_2}{4}.$$

This immediately gives (2.50) for $L \ge L_1$. It then follows for all $L > 0$ by obvious monotonicity in L.

Now fix $n = n(L)$ so that (2.50) holds. We can then apply the strong Markov property with respect to the stopping time $k_1 2^{-n}$, where

$$k_1 = k_1(L) := \inf\{k \in \mathbb{Z} : P^{Y_{k2^{-n(L)}}}\{\sup_{s < T_1} N_B(\mathcal{C}^c(L), s) \ge 1\} \ge \frac{\varepsilon_1}{4}\}.$$

We then obtain for all L

$$P^\sigma\{\sup_{s \le T_1+T_2+1} N_B(\mathcal{C}^c(L), s) \ge 1\} \ge P\{k_1 \le T_2 2^{n(L)+1}\}\frac{\varepsilon_1}{4} \ge \frac{\varepsilon_1\varepsilon_2}{8}. \tag{2.55}$$

But any particle ρ which is of type B at time $T_1 + T_2 + 1$ has moved at most over the distance

$$\sup_{s_1, s_2 \le T_1+T_2+1} \|\pi_B(s_1, \rho) - \pi_B(s_2, \rho)\|$$

since its creation (see (2.10)). This displacement is finite a.s. $[P^\sigma]$. However, (2.55) says that there exist B-particles arbitrarily far out with probability at least $\varepsilon_1\varepsilon_2/8$. Therefore, it must be the case that

$$P^\sigma\{\text{infinitely many } B\text{-particles are created during } [0, T_1 + T_2 + 1]\} \ge \frac{\varepsilon_1\varepsilon_2}{8}.$$

This, however, is impossible for $\sigma \in \Sigma_0$ (see (2.18)). Thus (2.49) leads to a contradiction and (2.48) must hold.

Step 3. Note that (2.48), together with the Borel-Cantelli lemma, implies that a.s. $[P^\sigma]$

$$\lim_{L\to\infty} P^{Y_t}\{\mathcal{H}(1, L, T)\} = 0 \text{ for all } t \ge 0. \tag{2.56}$$

We now show that this, together with (2.45) and (2.36), implies the important continuity property (2.43). From there it is easy to complete the proof of the proposition. To see (2.43) let t be any time $\le T$ and take $\sigma_2 = Y_t, \sigma_1 = Y_{t+h}$ and apply Lemma 3. Note that the roles of σ_1 and σ_2 have been interchanged from their roles in the lines following (2.50). As before we obtain for $\mathcal{E}$ a finite union of sets of the form (2.34) that for $K \ge L$, $K \ge$ some $K_1(\mathcal{E})$ and sufficiently small $h > 0$

$$\begin{aligned}&\big|P^{Y_{t+h}}\{\mathcal{E}\} - P^{Y_t}\{\mathcal{E}\}\big| \\ &\le P^{Y_t}\{\mathcal{H}(1, L, T)\} + P^{Y_{t+h}}\{\mathcal{H}(2, L, K, T)\} + 2P^{Y_t}\{\mathcal{H}(2, L, K, T)\} \\ &+ I[Y_s \text{ has some } B\text{-particles outside } \mathcal{C}(L) \text{ for some } s \le T + 1].\end{aligned} \tag{2.57}$$

We now first let $h \downarrow 0$ to obtain

$$\begin{aligned}\limsup_{h\downarrow 0} P^{Y_{t+h}}\{\mathcal{E}\} &\le P^{Y_t}\{\mathcal{E}\} + P^{Y_t}\{\mathcal{H}(1, L, T)\} \\ &+ \limsup_{h\downarrow 0} P^{Y_{t+h}}\{\mathcal{H}(2, L, K, T)\} + 2P^{Y_t}\{\mathcal{H}(2, L, K, T)\} \\ &+ I[Y_s \text{ has some } B\text{-particles outside } \mathcal{C}(L) \text{ for some } s \le T + 1].\end{aligned}$$

Next we let $K \to \infty$ and then $L \to \infty$ and apply (2.45),(2.56) and (2.54), respectively. This yields that a.s. $[P^\sigma]$

$$\limsup_{h\downarrow 0} P^{Y_{t+h}}\{\mathcal{E}\} \le P^{Y_t}\{\mathcal{E}\} \text{ for all } t \ge 0.$$

The opposite inequality for liminf instead of limsup is proven in the same way, so that (2.43) is proven.

We come now to (2.42). We already proved the first inequality in (2.51). Moreover, we already know from Lemma 2 that a.s. $[P^\sigma]$, $M_s(Y_t) < \infty$ for all $s, t \geq 0$. As in the proof of (2.20) this shows that each Y_t has the property that

$$P^{Y_t}\{\sup_{u \leq T} \big(\text{number of } A\text{-particles at } (z, u)\big) < \infty\} = 1 \tag{2.58}$$

for all $z \in \mathbb{Z}^d$ and $T \geq 0$. Consequently, starting in the state Y_t, only finitely many B-particles are created at any τ_k and therefore, a.s. $[P^{Y_t}]$, $\tau_{k+1} > \tau_k$ for all k and one cannot have $\widehat{\tau} < \tau_\infty$. Therefore we only have to prove that a.s. $[P^\sigma]$ it holds for all s, t that $P^{Y_t}\{\tau_\infty \leq s\} = 0$. Now assume that this fails. Then there exists some $C > 0$, $1 < D, s < \infty$ such that

$$P^\sigma\Big\{P^{Y_t}\{\min\{\widehat{\tau}, \tau_\infty\} \leq s\} \geq C \text{ for some } t \leq D - 1\Big\} \geq C. \tag{2.59}$$

Consequently,if $\rho_1, \rho_2, \dots$ is some ordering of all particles in σ, then for any E, there exist $F = F(E), G = G(E) < \infty$ for which

$$\begin{aligned} P^\sigma\Big\{P^{Y_t}\{&Y_s''(\rho) = B, Y_s'(\rho) \in \mathcal{C}(F) \text{ for at least } E \text{ particles} \\ &\text{among } \rho_1, \dots, \rho_G\} \geq \frac{C}{2} \\ &\text{for some } t \leq D - 1\Big\} \geq \frac{C}{2}. \end{aligned} \tag{2.60}$$

But by (2.43) the P^{Y_t}-probability appearing here is right continuous in t. Thus there exists an $n < \infty$ such that

$$\begin{aligned} P^\sigma\Big\{P^{Y_{k2^{-n}}}\{&Y_s''(\rho) = B, Y_s'(\rho) \in \mathcal{C}(F) \text{ for at least } E \text{ particles} \\ &\text{among } \rho_1, \dots, \rho_G\} \geq \frac{C}{4} \\ &\text{for some } k \leq D2^n\Big\} \geq \frac{C}{4}. \end{aligned} \tag{2.61}$$

We then apply the strong Markov property with respect to the stopping time $k_2 2^{-n}$, where

$$\begin{aligned} k_2 := \inf\Big\{k : P^{Y_{k2^{-n}}}\{&Y_s''(\rho) = B, Y_s'(\rho) \in \mathcal{C}(F) \text{ for at least } E \text{ particles} \\ &\text{among } \rho_1, \dots, \rho_G\} \geq \frac{C}{4}\Big\}. \end{aligned}$$

This yields

$$\begin{aligned} &P^\sigma\{\nu(D + s) \geq E\} \\ &\quad = P^\sigma\{\text{there are at least } E \text{ particles of type } B \text{ at time } D + s\} \\ &\quad \geq P\{k_2 2^{-n} \leq D\}\frac{C}{4} \geq \frac{C^2}{16}, \end{aligned}$$

for all E. By taking $E \to \infty$ we therefore obtain

$$P^\sigma\{\tau_\infty \le D + s\} \ge \frac{C^2}{16},$$

and this contradicts (2.18). Thus (2.59) cannot hold and (2.42) must be true. This also proves (2.41), i.e., if the Y-process is started in a state $\sigma \in \Sigma_0$, then it stays in Σ_0 forever, a.s. $[P^\sigma]$.

Lastly, it follows from (2.43) that this process on Σ_0 has the strong Markov property. This can be seen by minor modifications of standard proofs of the strong Markov property for a Feller process (see [D], Theorem 5.10 and the remark following it, or [BG], Theorem I.8.11). ■

We have now shown that the Y-process on Σ_0 is a nice Markov process, but before we can accept it as a version of a process as described in the abstract we have to show that Σ_0 is not empty. In the next proposition we shall show the even stronger property that σ lies in Σ_0 a.s. if σ is chosen by putting $N_A(z, 0-)$ A-particles at z, with the $N_A(z, 0-)$ i.i.d. mean μ_A Poisson variables, and by adding in total a finite number of B-particles. *From now on P without superscript will be used for the measure governing the Y-process with such an initial measure.* This notation does not indicate the value of μ_A, nor the location of the B-particles introduced at time 0, but these quantities have no significant influence anyway. Expectation with respect to P will be denoted by E without superscript. Note that the description of our system in the abstract forces all particles at any given space-time point to be of the same type. Thus if we put B-particles at $z_1, \dots, z_k$ at time 0, then we instantaneously have to change the A-particles there into B-particles.

The proof of the next proposition is basically a Peierls argument. We associate to each B-particle present at time t and with a switching time before τ_∞ a different "genealogical path" which describes how the B-particle arose from the B-particles at time 0 by various coincidences between A and B-particles, and then more or less count all the genealogical paths to show that the expected number of genealogical paths at each time $t < \infty$ is finite.

Proposition 5. *For any choice of the location of the finite number of initial B-particles we have $\sigma \in \Sigma_0$ a.s. $[P]$. Equivalently*

$$\int P\{Y_0 \in d\sigma\} P^\sigma\{\min\{\widehat{\tau}, \tau_\infty\} = \infty\} = 1. \tag{2.62}$$

Proof. It is a trivial calculation to show that $EM_t < \infty$, so that

$$M_t(\sigma) < \infty \text{ for all integers } t \text{ simultaneously, a.s. } [P].$$

The restriction of t to the integers here can be dropped by means of (2.22). Exactly as in the proof of (2.42) this shows that $\widehat{\tau} \ge \tau_\infty$ a.s. $[P]$, so that we only have to show

$$P\{\tau_\infty = \infty\} = 1. \tag{2.63}$$

At any time t, we shall associate to any particle ρ that has type B, and turned into a B-particle strictly before τ_∞ a unique genealogical path. One component of the genealogical path is a space-time path $\widetilde{\zeta}(\cdot, \rho)$ on $[0,t]$ which keeps track of the space-time paths traversed by the "ancestors" of ρ, i.e., by B-particles which "transmitted" the B-type to ρ. The genealogical path also contains some additional information about the identity of the transmitting particles. The space-time part $\widetilde{\zeta}$ of the genealogical path is constructed as follows. If ρ starts at z and has type B already at time 0, then its genealogical path is just the space-time path followed by ρ, restricted to $[0,t]$, that is $\zeta(s,\rho) = \pi(s,\rho) = \pi(0,\rho) + \pi_B(s,\rho) = z + \pi_B(s,\rho)$, $0 \le s \le t$. If ρ initially has type A, then ρ first turned into a B-particle at its switching time $\theta(\rho)$, which necessarily is less than or equal to t. Moreover, we assumed ρ became of type B before τ_∞, i.e., $\theta(\rho) < \tau_\infty$. The path component of the genealogical path of ρ will then be $\widetilde{\zeta}(s,\rho) = \pi(s,\rho)$ for $\theta(\rho) \le s \le t$. Note $\theta(\rho) = \tau_k$ for some k. At this time either the particle ρ jumped from some site y to a site which contained some B-particle ρ', or there was a B-particle ρ' which jumped from some position y onto the position of ρ at time $\theta(\rho)$. In the former case ρ' may not be unique, but we make some choice for ρ' among the B-particles at the site to which ρ jumps at time $\theta(\rho)$. We now follow the particle ρ' backwards in time till time $\theta(\rho')$ when it first turned into a B-particle, and take $\zeta(s,\rho) = \pi(s,\rho')$ for $\theta(\rho') \le s < \theta(\rho)$. If $\theta(\rho') = 0$, then we have defined the genealogical path of ρ on the whole interval $[0,t]$ and we are done. If $\theta(\rho') > 0$, then $\theta(\rho') = \tau_{k'}$ for some $k' < k$ and ρ' coincided with some other B-particle ρ'' at $\theta(\rho')$. We then follow ρ'' backwards in time etc., till we arrive at time 0. If we now traverse $\widetilde{\zeta}$ in the natural direction from 0 to t, then we see that this path starts with following the path of some initial B-particle, ρ_0, till some time s_1. At time s_1 either ρ_0 jumps to a point where there is an A-particle ρ_1, or some A-particle ρ_1 jumps at time s_1 to the position of ρ_0 at time s_1. Thus ρ_1 turns into a B-particle at time s_1, so that $s_1 = \theta(\rho_1)$. The path $\widetilde{\zeta}$ then follows the path of ρ_1 till some time s_2, at which ρ_1 coincides with an A-particle ρ_2, which turns into a B-particle due to this coincidence. This continues until some time s_ℓ, at which the A-particle ρ turns into a B-particle. Thus s_ℓ equals what we called $\theta(\rho)$ before. $\widetilde{\zeta}$ then equals $\pi(\cdot,\rho)$ on $[s_\ell, t]$. We shall take ρ_ℓ equal to ρ.

We shall want to keep track of some further data in the genealogical path. It will be convenient at this stage to label the initial A-particles by their initial position and their number in some arbitrary ordering of the initial particles at that site. Thus $\langle z, m\rangle$ will be used to denote the m-th particle which started at z. We shall say that the particle $\langle z, m\rangle$ *exists* if and only if there are at least m particles at z at time 0. The particle ρ_i appearing in the genealogical path in the preceding paragraph will also be denoted by $\langle z_i, m_i\rangle$. We denote by η_i the type of the particle which jumps at time s_i. This particle can be ρ_{i-1} or ρ_i. η_i takes one of the values A or B. We further denote for $1 \le i \le \ell$ by y_i the position from which the particle jumps at time s_i, and by x_i the position to which the particle jumps at time s_i. $x_{\ell+1}$ will be the position of $\rho = \rho_\ell$ at time t. The full genealogical path associated to ρ now consists of $\widetilde{\zeta}(\cdot,\rho)$ plus the $(x_i, s_i, \eta_i, y_i, \rho_0, \rho_i = \langle z_i, m_i\rangle)$, that is, the positions and times of the jumps at which there is a changeover from one particle to another, as well as which particles jump at these times and which particles continue along $\widetilde{\zeta}(\cdot, \langle z_i, m_i\rangle)$. We obtain the genealogical paths of all B-particles at time t by this forward construction and taking all possible values of $\widetilde{\zeta}, \ell$ and $(x_i, s_i, \eta_i, y_i, \rho_0, \rho_i), 1 \le i \le \ell$. Since each genealogical path is the genealogical path of just one particle, namely ρ_ℓ, the number of B-particles at time t is at most equal to the number of genealogical paths obtained in this forward construction. A crucial observation is that the ρ_i, $0 \le i \le \ell$, *have to be distinct*. Indeed, in the construction of the genealogical path, ρ_i is a particle whose type changes from A to B at time s_i (with $s_0 = 0$), and any particle can change from type A to type B only once. We also note that ρ_i becomes a B-particle at time s_i and then must move from x_i to x_{i+1} during $[s_i, s_{i+1})$ if $\eta_{i+1} = A$, or must move from x_i to y_{i+1} during $[s_i, s_{i+1})$ if $\eta_{i+1} = B$.

We claim that it suffices for (2.63) to prove that for any t

$$E\{\text{number of genealogical paths associated to some } B\text{-particle at time } t\} < \infty. \tag{2.64}$$

Indeed, this will imply that the number of B-particles which arises before t is almost surely finite. Since infinitely many B-particles have been created by time τ_∞ this will also give

$$P\{\tau_\infty < t\} = 0 \text{ for any } t. \tag{2.65}$$

Thus (2.64) is indeed sufficient for (2.63). For the time being we shall estimate

$$\begin{aligned} E\{&\text{number of genealogical paths associated to some } B\text{-particle} \\ &\qquad\qquad\qquad\text{that is in the set } E \text{ at time } t\}, \end{aligned} \tag{2.66}$$

for any subset E of $\mathbb{Z}^d$. Only near the end of this proof shall we take $E = \mathbb{Z}^d$ to get (2.63).

We bound the expectation in (2.66) by decomposing with respect to ℓ and the data $(x_i, s_i, \eta_i, y_i, \rho_0, \rho_i = \langle z_i, m_i\rangle), 1 \le i \le \ell$. Of course we cannot directly decompose with respect to the s_i, but have to follow the usual procedure which specifies only that the jump occurs in some interval $J(k) = J_n(k) := (k/n, (k+1)/n]$ and then let n go to infinity. To this end we introduce the following indicator functions (with $\mathbf{z}$ used as an abbreviation for an ℓ-tuple $z_1, \dots, z_\ell$, and similarly for $\mathbf{m}, \mathbf{k}$): if $\eta_i = A$ and $2 \le i \le \ell$, then

$$\begin{aligned} I_i(\mathbf{k}, \mathbf{z}, \mathbf{m}) = I_{i,A}(\mathbf{k}, \mathbf{z}, \mathbf{m}) = I\big[&\langle z_{i-1}, m_{i-1}\rangle \text{ is at } x_{i-1} \text{ at time } (k_{i-1}+1)/n \\ &\text{and moves from there to } x_i \text{ during } [(k_{i-1}+1)/n, k_i/n)\big] \\ &\times I\big[\langle z_i, m_i\rangle \text{ is an } A\text{-particle at } y_i \text{ at time } (k_i/n)-, \\ &\text{it jumps to } x_i \text{ during } J(k_i) \text{ and becomes the } B\text{-particle } \rho_{i+1}\big], \end{aligned} \tag{2.67}$$

whereas for $\eta_i = B,\ 2 \le i \le \ell$,

$$\begin{aligned} I_i(\mathbf{k},\mathbf{z},\mathbf{m}) = I_{i,B}(\mathbf{k},\mathbf{z},\mathbf{m}) = I\big[&\langle z_{i-1}, m_{i-1}\rangle \text{ is at } x_{i-1} \text{ at time } (k_{i-1}+1)/n \\ &\text{and moves from there to } y_i \text{ during } [(k_{i-1}+1)/n, k_i/n)\big] \\ \times I\big[&\langle z_i, m_i\rangle \text{ is an } A\text{-particle at } x_i \text{ at time } (k_i/n)- \text{ and} \\ &\text{during } J(k_i) \text{ the } B\text{-particle } \langle z_{i-1}, m_{i-1}\rangle \text{ jumps from } y_i \text{ to } x_i\big]; \end{aligned} \tag{2.68}$$

$$I_{\ell+1} = I\big[\rho_\ell \text{ moves to a position in } E \text{ during } [(k_\ell+1)/n, t]. \tag{2.69}$$

For $i = 1$ the definitions of $I_{1,A}$ and $I_{1,B}$ need small changes, which amount to interpreting $\langle z_0, m_0\rangle$ as ρ_0, k_0 as -1 and x_0 as the initial position of ρ_0. For instance, $I_{1,B}$ is defined as

$$\begin{aligned} I_{1,B} = I_{1,B}(\mathbf{k},\mathbf{z},\mathbf{m}) &= I\big[\rho_0 \text{ moves from } x_0 \text{ to } y_1 \text{ during } [0, k_1/n)\big] \\ \times I\big[&\langle z_1, m_1\rangle \text{ is an } A\text{-particle at } x_1 \text{ at time } (k_1/n)- \text{ and} \\ &\text{during } J(k_i) \text{ the } B\text{-particle } \rho_0 \text{ jumps from } y_1 \text{ to } x_1\big]. \end{aligned} \tag{2.70}$$

We leave the corresponding definition of $I_{1,A}$ to the reader. Finally we define

$$\begin{aligned} H = H_n(\mathbf{k},\mathbf{z},\mathbf{m}) = I\big[&\text{the particles } \rho_j, 1 \le j \le \ell, \text{ together} \\ &\text{have at most one jump during } J_n(k_i),\ 1 \le i \le \ell\big]. \end{aligned} \tag{2.71}$$

We shall use $\prod^{(\eta)}$ to denote the product over the indices $i \in [1,\ell]$ with $\eta_i = \eta$. Also $\sum^{(\ell)}$ is the sum over all ℓ-tuples $\rho_1, \cdots, \rho_\ell$ of initial A-particles which are distinct, and distinct from ρ_0. Finally, $\sum_{x_1,\dots,x_{\ell+1}}$ will be short for the sum over $x_1, \dots, x_\ell \in \mathbb{Z}^d$, and over $x_{\ell+1} \in E$.

We claim that for fixed ρ_0 and $\mathbf{z}, \mathbf{m}$ there are almost surely no common jump times in the paths of any pair of particles from $\{\rho_0, \rho_i = \langle z_i, m_i\rangle : 1 \le i \le \ell\}$. This follows from the fact that for any ρ, $\pi(\cdot,\rho)$ can have a jump at a time s only if $\pi_A(\cdot,\rho)$ or $\pi_B(\cdot,\rho)$ has a jump at s (by virtue of (2.2), (2.4)). Our claim then follows because all the paths $\pi_A(\cdot,\rho_i)$ and $\pi_B(\cdot,\rho_i)$ for different i are independent. It follows that, for given $\mathbf{z},\mathbf{m}$, we have almost surely $\inf_{\mathbf{k}} H_n(\mathbf{k},\mathbf{z},\mathbf{m}) \to 1$ as $n \to \infty$. Consequently,

$$\begin{aligned} \sum_{\rho_0} \sum_{x_1,\dots,x_{\ell+1}} \sum_{\eta_1,\dots,\eta_\ell} \sum_{y_1,\dots,y_\ell} \sum_{0<k_1<\cdots<k_\ell<nt} & H_n I_{\ell+1} \\ \times \prod{}^{(A)} I_{i,A}(z_i,m_i) & \prod{}^{(B)} I_{i,B}(z_i,m_i) \end{aligned}$$

almost surely converges as $n \to \infty$ to the number of genealogical paths with exactly ℓ changeovers among given pairs $\langle z_{i-1}, m_{i-1}\rangle$ and $\langle z_i, m_i\rangle$, and with final position in E at time t. Thus, by Fatou's lemma,

$$
\begin{aligned}
&E\{\text{number of genealogical paths with } \ell \text{ changeover times associated to}\\
&\qquad\qquad \text{some } B\text{-particle which is in } E \text{ at time } t\}\\
&\le \liminf_{n\to\infty} E\Big\{\sum_{\rho_0}\sum_{x_1,\dots,x_{\ell+1}}\sum_{\eta_1,\dots,\eta_\ell}\sum_{y_1,\dots,y_\ell}\sum_{0<k_1<\dots<k_\ell<nt}\sum{}^{(\ell)} H_n I_{\ell+1}\\
&\qquad\qquad \times \prod{}^{(A)} I_{i,A}(z_i,m_i)\prod{}^{(B)} I_{i,B}(z_i,m_i)\Big\}. \qquad (2.72)
\end{aligned}
$$

Note also that our particles perform simple random walks, so that the sum over y_i in (2.72) can be restricted to the neighbors of x_i. The sum over ρ_0 runs over the finite number of initial B-particles. For simplicity we restrict ourselves in the remainder of this proof to the case in which there is only one initial B-particle, and that it starts from position x_0. We can then drop the sum over ρ_0.

We wish to establish some independence between the required jumps and the required movement of B-particles in the indicator functions in the right hand side of (2.72). For this we shall again make use of the particle system $\mathcal{P}^*$, which we coupled to our true particle system just before Lemma 2. We shall use $\mathcal{P}_0$ to denote the true particle system and use $N_A(x,t)$ for the number of A particles at the space-time point (x,t) in this true system. Recall that we coupled $\mathcal{P}^*$ and $\mathcal{P}_0$ in such a way that $N^*(x,0) = N_A(x,0-)$ for all x. Thus in the present situation the $N^*(x,0)$ are i.i.d. mean μ_A Poisson variables. According to our construction $N_A(x,t) = 0$ for $x \in \mathbb{Z}^d, t \ge \tau_\infty$, and

$$
N_A(x,t) \le N^*(x,t) \text{ for all } x \in \mathbb{Z}^d, t \ge 0. \qquad (2.73)
$$

It also follows that if $\eta_i = A$, then

$$
\begin{aligned}
&I_{i,A}(\mathbf{k},\mathbf{z},\mathbf{m})\\
&\quad \le I\big[\text{in } \mathcal{P}_0,\ \langle z_{i-1},m_{i-1}\rangle \text{ moves from } x_{i-1} \text{ to } x_i \text{ during } [(k_{i-1}+1)/n, k_i/n)\big]\\
&\quad \times I\big[\text{in } \mathcal{P}^*,\ \langle z_i,m_i\rangle \text{ is at } y_i \text{ at time } (k_i/n)- \text{ and jumps to } x_i \text{ during } J(k_i)\big]\\
&\quad \le I\big[\langle z_i,m_i\rangle \text{ exists}\big] I\big[\pi_A(k_i/n,\langle z_i,m_i\rangle) = y_i - z_i \text{ and } \langle z_i,m_i\rangle\\
&\qquad\qquad\qquad\qquad \text{jumps to } x_i \text{ during } J(k_i)\big]\\
&\quad \times I\big[\pi_B\big(k_i/n,\langle z_{i-1},m_{i-1}\rangle\big) - \pi_B\big((k_{i-1}+1)/n,\langle z_{i-1},m_{i-1}\rangle\big) = x_i - x_{i-1}\big]\\
&\qquad =: K_{i,A}(\mathbf{k},\mathbf{z},\mathbf{m}) L_{i,A}(\mathbf{k},\mathbf{z},\mathbf{m}) \qquad (2.74)
\end{aligned}
$$

with $K_{i,A}$ standing for $I[\langle z_i,m_i\rangle \text{ exists}]$ times the indicator function involving $\pi_A(\cdot,\langle z_i,m_i\rangle)$, while $L_{i,A}$ stands for the indicator function involving $\pi_B(\cdot,\langle z_{i-1},m_{i-1}\rangle)$ in the right hand side. For $i=1$ we interpret k_0 as -1. Similarly, if $\eta_i = B$, then

$$
\begin{aligned}
&I_{i,B}(\mathbf{k},\mathbf{z},\mathbf{m})\\
&\quad \le I\big[\langle z_i,m_i\rangle \text{ exists}\big] I\big[\pi_A(k_i/n,\langle z_i,m_i\rangle) = x_i - z_i\big]\\
&\quad \times I\big[\pi_B\big(k_i/n,\langle z_{i-1},m_{i-1}\rangle\big) - \pi_B\big((k_{i-1}+1)/n,\langle z_{i-1},m_{i-1}\rangle\big) = y_i - x_{i-1}\\
&\qquad\qquad\qquad\qquad \text{and jumps to } x_i - x_{i-1} \text{ during } J(k_i)\big]\\
&\quad =: K_{i,B}(\mathbf{k},\mathbf{z},\mathbf{m}) L_{i,B}(\mathbf{k},\mathbf{z},\mathbf{m}). \qquad (2.75)
\end{aligned}
$$

For $i = 1$ we again take $k_0 = -1$. Lastly

$$I_{\ell+1} \le I\big[\pi_B\big(t, \langle z_\ell, m_\ell\rangle\big) - \pi_B\big(k_\ell + 1)/n, \langle z_\ell, m_\ell\rangle\big) \in E - x_\ell\big] =: L_{\ell+1}. \tag{2.76}$$

We may therefore replace $I_{i,A}, I_{i,B}, I_{\ell+1}$ in the right hand side of (2.72) by the appropriate right hand sides in (2.74)-(2.76). For brevity we drop $\mathbf{k}$ from the notation. By construction all the paths π_B and all the paths π_A are independent. It follows from this that the left hand side of (2.72) is bounded by

$$\begin{aligned}\liminf_{n\to\infty} \sum_{x_1,\dots,x_{\ell+1}} \sum_{\eta_1,\dots,\eta_\ell} \sum_{y_1,\dots,y_\ell} \sum_{0<k_1<\dots<k_\ell<nt} \sum\nolimits^{(\ell)} \\ E\Big\{H_n^A \prod\nolimits^{(A)} K_{i,A}(\mathbf{z},\mathbf{m}) \prod\nolimits^{(B)} K_{i,B}(\mathbf{z},\mathbf{m})\Big\} \\ \times E\Big\{H_n^B L_{\ell+1} \prod\nolimits^{(A)} L_{i,A}(\mathbf{z},\mathbf{m}) \prod\nolimits^{(B)} L_{i,B}(\mathbf{z},\mathbf{m})\Big\},\end{aligned} \tag{2.77}$$

where

$$H_n^\eta = I[\text{each particle } \rho_j \text{ with } \eta_j = \eta \text{ has exactly one jump in } J(k_j)].$$

With S^η as in the beginning of this section we can write

$$\begin{aligned}&E\Big\{H_n^B L_{\ell+1} \prod\nolimits^{(A)} L_{i,A}(\mathbf{z},\mathbf{m}) \prod\nolimits^{(B)} L_{i,B}(\mathbf{z},\mathbf{m})\Big\} \\ &\le \prod\nolimits^{(A)} P\{S^B_{(k_i-k_{i-1}-1)/n} = x_i - x_{i-1}] \\ &\times \prod\nolimits^{(B)} P\{S^B_{(k_i-k_{i-1}-1)/n} = y_i - x_{i-1} \text{ and has exactly one jump during} \\ &\qquad J(k_i) \text{ and this goes from } y_i - x_{i-1} \text{ to } x_i - x_{i-1}\} \\ &\times P\{S^B_{t-(k_\ell+1)/n} \in E - x_\ell\}.\end{aligned} \tag{2.78}$$

To simplify our formulae somewhat we now use that, as in (2.22),

$$\begin{aligned}&P\{S^B_{(k_i-k_{i-1})/n} = x_i - x_{i-1}\} \\ &\ge P\{S^B_{(k_i-k_{i-1}-1)/n} = x_i - x_{i-1}\} P\{S^B_\cdot \text{ remains constant} \\ &\qquad \text{during } [(k_i - k_{i-1} - 1)/n, (k_i - k_{i-1})/n]\} \\ &= e^{-D_B/n} P\{S^B_{(k_i-k_{i-1}-1)/n} = x_i - x_{i-1}\}.\end{aligned}$$

We write $\nu = \nu(\eta) = \nu(\eta, \ell)$ for the number of $1 \le i \le \ell$ with $\eta_i = A$. Then the last inequality combined with (2.78) shows that

$$\begin{aligned}&E\Big\{H_n^B L_{\ell+1} \prod\nolimits^{(A)} L_{i,A}(\mathbf{z},\mathbf{m}) \prod\nolimits^{(B)} L_{i,B}(\mathbf{z},\mathbf{m})\Big\} \\ &\le e^{\nu D_B/n} P\{S^B_t \in E - x_0, S^B_{k_i/n} = x_i - x_0, \text{ for } \eta_i = A; S^B_{k_i/n} = y_i - x_0 \text{ and} \\ &\qquad S^B_\cdot \text{ jumps from } y_i - x_0 \text{ to } x_i - x_0 \text{ during } J(k_i) \text{ for } \eta_i = B\}.\end{aligned} \tag{2.79}$$

The right hand side is independent of $\mathbf{z}, \mathbf{m}$, so the expectation of the L factors in the right hand side of (2.77) can be taken outside the sum $\sum^{(\ell)}$.

Next we deal with $\sum^{(\ell)}$ of the expectation of the factors K_i. We claim that

$$\sum_{m_1,\dots,m_\ell} E\Big\{H_n^A \prod{}^{(A)} K_{i,A}(\mathbf{z},\mathbf{m}) \prod{}^{(B)} K_{i,B}(\mathbf{z},\mathbf{m})\Big\}$$
$$\leq \Big[\frac{D_A \mu_A}{2dn}\Big]^\nu \prod{}^{(A)} P\{S^A_{k_i/n} = y_i - z_i\} \prod{}^{(B)} P\{S^A_{k_i/n} = x_i - z_i\}, \tag{2.80}$$

provided $\sum_{m_1,\dots m_\ell}$ runs only over those ℓ-tuples with $m_i \geq 1$ for which $\langle z_i, m_i\rangle$, $1 \leq i \leq \ell$, are distinct. To prove this, first fix $\mathbf{z}, \mathbf{m}$ such that all $\langle z_i, m_i\rangle$, $1 \leq i \leq \ell$, are distinct A-particles (and distinct from ρ_0). For such ℓ-tuples, the paths $\pi_A(\cdot, \langle z_i, m_i\rangle)$ are independent, and

$$P\{\pi_A(k_i/n, \langle z_i, m_i\rangle) = y_i - z_i \text{ and } \langle z_i, m_i\rangle \text{ jumps to } x_i \text{ during } J(k_i)\}$$
$$\leq \frac{D_A}{2dn} P\{S^A_{k_i/n} = y_i - z_i\},$$

while

$$P\{\pi_A(k_i/n, \langle z_i, m_i\rangle) = x_i - z_i\} = P\{S^A_{k_i/n} = x_i - z_i\}.$$

Consequently, the left hand side of (2.80) is at most

$$\sum_{m_1,\dots,m_\ell} E\Big\{\prod_{i=1}^{\ell} I[\langle z_i, m_i\rangle \text{ exists}]\Big\}$$
$$\times \Big[\frac{D_A}{2dn}\Big]^\nu \prod{}^{(A)} P\{S^A_{k_i/n} = y_i - z_i\} \prod{}^{(B)} P\{S^A_{k_i/n} = x_i - z_i\}.$$

Therefore it suffices for (2.80) to show that

$$E\Big\{\sum_{m_1,\dots,m_\ell} \prod_{i=1}^{\ell} I[\langle z_i, m_i\rangle \text{ exists}]\Big\} \leq [\mu_A]^\nu. \tag{2.81}$$

To prove this last inequality, we partition the z_i into maximal classes of equal z's. More precisely, let $a_1, \dots, a_p \in \mathbb{Z}^d$ be distinct, and let $T_1, \dots, T_p$ be a partition of $\{1, \dots, \ell\}$ and let $z_i = a_j$ precisely for $i \in T_j$. Finally, let T_j have exactly q_j elements. If we write $[N]_k$ for $N(N-1)\cdots(N-k+1)$, then

$$\sum_{m_1,\dots,m_\ell} \prod_{i=1}^{\ell} I[\langle z_i, m_i\rangle \text{ exists}] = \prod_{j=1}^{p} \big[N_A(a_j, 0)\big]_{q_j}. \tag{2.82}$$

The inequality (2.81) now follows by taking the expectation in (2.82). (In fact, since we assumed that the N_A have a Poisson distribution, (2.81) holds with equality.

We point out here that (2.81) also holds if $N_A(z,0) \le \mu_A$ with probability 1, rather than distributed like a mean μ_A Poisson variable.)

As pointed out, (2.81) proves (2.80). If we sum (2.80) over the z_i and use (2.79) we obtain

$$\begin{aligned}\sum{}^{(\ell)} E\Big\{H_n^A \prod{}^{(A)} K_{i,A}(z_i,m_i) \prod{}^{(B)} K_{i,B}(z_i,m_i)\Big\}\\ \times E\Big\{H_n^B L_{\ell+1} \prod{}^{(A)} L_{i,A}(z_i,m_i) \prod{}^{(B)} L_{i,B}(z_i,m_i)\Big\}\\ \le \Big[\frac{D_A\mu_A e^{D_B/n}}{2dn}\Big]^\nu P\{S_t^B \in E - x_0, S^B_{k_i/n} = x_i - x_0, \text{ for } \eta_i = A; S^B_{k_i/n} = y_i - x_0\\ \text{and } S^B_\cdot \text{ jumps from } y_i - x_0 \text{ to } x_i - x_0 \text{ during } J(k_i) \text{ for } \eta_i = B\}. \end{aligned} \tag{2.83}$$

We now fix the set of indices for which $\eta_i = B$. Let this set be $\mathcal{D} = \{i_1 < i_2 < \cdots < i_\kappa\} \subset \{1,\dots,\ell\}$. We also fix the k_{i_j} for $1 \le j \le \kappa$. Note that $\mathcal{D} = \emptyset$, or equivalently, $\kappa = 0$ is possible. Further set $i_0 = 0, i_{\kappa+1} = \ell+1, k_0 = -1, k_{\ell+1} = \lfloor nt \rfloor$. Finally note that

$$\nu = \ell - \kappa = \sum_{j=0}^{\kappa} [i_{j+1} - i_j - 1], \tag{2.84 }$$

and that for all integers $a, b, r \ge 0$

$$\sum_{a < k_{p+1} < k_{p+2} < \cdots < k_{p+r} \le b} 1 = \binom{b-a}{r} \le \frac{(b-a)^r}{r!} \tag{2.85}$$

(the sum here is over $k_{p+1},\dots,k_{p+r}$). We now sum (2.83) first over all x_i, y_i with $i \notin \mathcal{D}$. Next sum over the k_j with $j \ge 1$, but $j \notin \mathcal{D}$, making use of (2.85). In this way we obtain that the contribution to (2.72) of the terms with $\eta_i = B$ exactly for $i \in \mathcal{D} = \{i_1 < i_2 < \cdots < i_\kappa\}$, with $\ell, \mathcal{D}$ and k_{i_j} for $i_j \in \mathcal{D}$ fixed (before taking the liminf over n), is at most

$$\begin{aligned}&\sum_{x_i, y_i,\, i \in \mathcal{D}} \Big[\frac{D_A\mu_A e^{D_B/n}}{n}\Big]^{\ell-\kappa} \frac{\big(k_{i_1}\big)^{i_1-1}}{(i_1-1)!} \prod_{j=1}^{\kappa} \frac{\big(k_{i_{j+1}} - k_{i_j}\big)^{i_{j+1}-i_j-1}}{\big(i_{j+1}-i_j-1\big)!}\\ &\times P\{S_t^B \in E - x_0, \text{ and } S^B_\cdot \text{ jumps from } y_i - x_0 \text{ to } x_i - x_0 \text{ during } J(k_i) \text{ for } i \in \mathcal{D}\}\\ &= \Big[\frac{D_A\mu_A e^{D_B/n}}{n}\Big]^{\ell-\kappa} \frac{\big(k_{i_1}\big)^{i_1-1}}{(i_1-1)!} \prod_{j=1}^{\kappa} \frac{\big(k_{i_{j+1}} - k_{i_j}\big)^{i_{j+1}-i_j-1}}{\big(i_{j+1}-i_j-1\big)!}\\ &\times P\{S_t^B \in E - x_0, \text{ and } S^B_\cdot \text{ has a jump during } J(k_i) \text{ for } i \in \mathcal{D}\}\\ &= \frac{1}{(i_1-1)!}\Big(\frac{D_A\mu_A k_{i_1} e^{D_B/n}}{n}\Big)^{i_1-1}\\ &\times \prod_{j=1}^{\kappa}\Big[\frac{1}{\big(i_{j+1}-i_j-1\big)!}\Big(\frac{D_A\mu_A\big(k_{i_{j+1}} - k_{i_j}\big)e^{D_B/n}}{n}\Big)^{i_{j+1}-i_j-1}\Big]\\ &\times P\{S_t^B \in E - x_0, \text{ and } S^B_\cdot \text{ has a jump during } J(k_i) \text{ for } i \in \mathcal{D}\}. \end{aligned} \tag{2.86}$$

We now sum (2.72) also over $\ell \geq \kappa$ and use Fatou's lemma to bring the liminf outside the sum over ℓ. We also rename k_{i_j} as r_j. Since $i_{\kappa+1} = \ell+1$ and $r_{\kappa+1} = \lfloor nt \rfloor$ this yields

$$\begin{aligned}
&E\{\text{number of genealogical paths associated to} \\
&\qquad\qquad\qquad\text{some } B\text{-particle which is in } E \text{ at time } t\} \\
&\leq \liminf_{n\to\infty} \sum_{\kappa \geq 0} \sum_{\mathcal{D}=\{i_1<\cdots<i_\kappa\}} \sum_{0<r_1<\cdots<r_\kappa \leq nt} \exp\Big[\frac{D_A\mu_A e^{D_B/n}}{n}\big(\lfloor nt \rfloor - r_\kappa\big)\Big] \\
&\quad \times \frac{1}{(i_1-1)!}\Big(\frac{D_A\mu_A r_1 e^{D_B/n}}{n}\Big)^{i_1-1} \\
&\quad \times \prod_{j=1}^{\kappa-1}\Big[\frac{1}{(i_{j+1}-i_j-1)!}\Big(\frac{D_A\mu_A\big(r_{j+1}-r_j\big)e^{D_B/n}}{n}\Big)^{i_{j+1}-i_j-1}\Big] \\
&\quad \times P\{S_t^B \in E - x_0, \text{ and } S_\cdot^B \text{ has a jump during } J(r_j) \text{ for } 1 \leq j \leq \kappa\ \}.
\end{aligned} \tag{2.87}$$

We next carry out the sum over $\mathcal{D} = \{i_1 < \cdots < i_\kappa\}$. This transforms the right hand side of (2.87) into

$$\begin{aligned}
&\liminf_{n\to\infty} \sum_{\kappa \geq 0} \sum_{0<r_1<\cdots<r_\kappa \leq nt} \exp\Big[\frac{D_A\mu_A r_1 e^{D_B/n}}{n} + \sum_{j=1}^{\kappa} \frac{D_A\mu_A(r_{j+1}-r_j)e^{D_B/n}}{n}\Big] \\
&\qquad \times P\{S_t^B \in E - x_0, \text{ and } S_\cdot^B \text{ has a jump during } J(r_j) \text{ for } 1 \leq j \leq \kappa\ \} \\
&= \liminf_{n\to\infty} \sum_{\kappa \geq 0} \sum_{0<r_1<\cdots<r_\kappa \leq nt} \exp\big[D_A\mu_A t\big] \\
&\qquad \times P\{S_t^B \in E - x_0, \text{ and } S_\cdot^B \text{ has a jump during } J(r_j) \text{ for } 1 \leq j \leq \kappa\ \}.
\end{aligned} \tag{2.88}$$

At this point we finally specialize to $E = \mathbb{Z}^d$. With this choice the right hand side of (2.88) is at most

$$\begin{aligned}
&\exp\big[D_A\mu_A t\big] \liminf_{n\to\infty} \sum_{\kappa \geq 0} \sum_{0<r_1<\cdots<r_\kappa \leq nt} \Big[\frac{D_B}{n}\Big]^\kappa \\
&\leq \exp\big[D_A\mu_A t\big] \liminf_{n\to\infty} \sum_{\kappa \geq 0} \frac{1}{\kappa!}\Big[\frac{D_B nt}{n}\Big]^\kappa \text{ (by (2.85))} \\
&= \exp\big[(D_A\mu_A + D_B)t\big] < \infty.
\end{aligned} \tag{2.89}$$

This proves (2.64) and the Proposition in the case when we start with one B-particle. If we start with N_B B-particles at $x_{0,1}, x_{0,2}, \dots, x_{0,N_B}$, respectively, then we only have to replace the probability in (2.88) by

$$\sum_{m=1}^{N_B} P\{S_t^B \in E - x_{0,m} \text{ and } S_\cdot^B \text{ has a jump during } J(r_j) \text{ for } 1 \leq j \leq \kappa\ \}. \tag{2.90}$$

(The $x_{0,m}$ do not have to be distinct here). ∎

Remark 2. A check of the proof shows that (2.81) is the only property of the initial distribution which is used. In particular, (2.63) also holds for any initial distribution for which $N_A(z,0)$ is a.s. bounded by a constant. A special case of this last situation is also treated in [RS], Lemma 3.2. The proof also works for initial distributions which are stochastically below a Poisson distribution. Also, by the argument given in the next section for Theorem 1, we obtain by specializing to $E = [\mathcal{C}(C_1t)]^c$, that (2.81) is sufficient to conclude that (1.3) holds.

3. A linear upper bound for $B(t)$.

In this section we give the **Proof of Theorem 1.** The arguments preceding (2.72) show that it is enough to show that for $E =$ the complement of $\mathcal{C}(C_1t)$, the left hand side of (2.72) is bounded by $2N_B\exp(-t)$. In turn, it suffices to prove that the right hand side of (2.88) (with the last factor replaced by (2.90)) is bounded by $2N_B\exp(-t)$ if we take $E = \mathcal{C}^c(C_1t)$. In order to show this we split the sum over κ in (2.88) into two pieces. The first sum is over $\kappa \ge K_1t$ and the second over $\kappa < K_1t$, where K_1 is chosen so large that the first piece is bounded by (compare (2.89))

$$\begin{aligned}
&\limsup_{n\to\infty} N_B \sum_{\kappa\ge K_1t}\ \sum_{0<r_1<\dots<r_\kappa\le nt} \exp\Big[\frac{D_A\mu_Ar_1e^{D_B/n}}{n} + \sum_{j=1}^{\kappa}\frac{D_A\mu_A(r_{j+1}-r_j)e^{D_B/n}}{n}\Big]\\
&\qquad\qquad \times P\{S^B_{\cdot} \text{ has a jump during } J(r_j) \text{ for } 1\le j\le\kappa\ \}\\
&\le N_B\exp\Big[D_A\mu_At\Big]\limsup_{n\to\infty}\sum_{\kappa\ge K_1t}\frac{1}{\kappa!}\Big[\frac{D_Bnt}{n}\Big]^\kappa\\
&\le N_Be^{-t}\ (\text{for } t\ge 1).
\end{aligned}\tag{3.1}$$

Note that this estimate is uniform in C_1.

To estimate the second sum, over $\kappa < K_1t$ (for fixed K_1), we note that the increments of S^B over disjoint intervals are independent. Thus the sum of the increments of S^B over

$$[0,t]\setminus\bigcup_{j=1}^{\kappa}J(r_j)$$

is the same as the distribution of

$$S^{B,0}_{r_1} + \sum_{j=1}^{\kappa-1}S^{B,j}_{(r_{j+1}-r_j-1)/n} + S^{B,\kappa+1}_{t-(r_\kappa+1)/n},$$

where the $S^{B,j}$ are independent copies of S^B. In turn, this sum has the same distribution as $S^B_{t-\kappa/n}$, and is independent of the increments of S^B over the $J(r_j)$. In addition, given that S^B has a jump in $J(r_j)$, the conditional distribution of $S^B_{(r_j+1)/n} - S^B_{r_j/n}$ is the distribution of

$$\sum_{m=1}^{\psi}Z_m,$$

where $Z_1, Z_2, \dots$ are independent random variables, each with the distribution of a generic jump of S^B, and ψ is independent of the Z_i, and ψ has the conditional distribution of a mean D_B/n Poisson variable, given that this variable is at least 1. In our case $P\{Z_i = \pm e_j\} = 1/(2d)$, so that $\big\| \sum_{m=1}^{\psi} Z_m \big\| \le \psi$, and conditionally on the event $\{S^B$ has a jump in $J(r_j), 1 \le j \le \kappa\}$, $\|S^B_t\|$ is stochastically smaller than

$$\|S^B_{t-\kappa/n}\| + \psi_1 + \dots + \psi_\kappa,$$

with the ψ_i independent copies of ψ, which are also independent of S^B. It is now a standard large deviation estimate that for fixed K_1 and $x_{0,m}$, and sufficiently large C_1 (independent of the $x_{0,m}$, though), and all sufficiently large t and $\kappa < K_1 t$

$$\begin{aligned} &P\{S^B_t \notin \mathcal{C}(C_1 t) - x_{0,m}, S^B_\cdot \text{ has a jump in } J(r_j),\ 1 \le j \le \kappa\} \\ &\le \Big[\frac{D_B}{n}\Big]^\kappa P\Big\{\|S^B_{t-\kappa/n}\| + \sum_{1 \le j < K_1 t} \psi_j \ge C_1 t/2\Big\} \le \Big[\frac{D_B}{n}\Big]^\kappa \exp[-(D_A\mu_A + D_B + 1)t]. \end{aligned}$$

We leave the details of this to the reader (compare (2.40) in [KS]). For such a choice of C_1 it follows that the sum of the terms with $\kappa < K_1 t$ in (2.88) (with the replacement of (2.90)) is at most

$$\begin{aligned} &\liminf_{n\to\infty} \sum_{1 \le m \le N_B} \sum_{0 \le \kappa < K_1 t} \sum_{0 < r_1 < \dots < r_\kappa \le nt} \exp\big[D_A\mu_A t\big] \\ &\qquad\qquad \times P\{S^B_t \in E - x_{0,m}, \text{ and } S^B_\cdot \text{ has a jump during } J(r_j) \text{ for } 1 \le j \le \kappa\ \} \\ &\le N_B \exp\big[D_A\mu_A t\big] \liminf_{n\to\infty} \sum_{0 \le \kappa < K_1 t} \frac{1}{\kappa!}\Big(\frac{ntD_B}{n}\Big)^\kappa \exp[-(D_A\mu_A + D_B + 1)t] \\ &\le N_B e^{-t}. \end{aligned}$$

For K_1, C_1 as above and $E = \mathcal{C}^c(C_1 t)$ we find that the expectation (2.66) is bounded by $2N_B \exp(-t)$ for all large t, so that (1.3) holds. (1.4) now follows from the Borel-Cantelli lemma and the fact that $B(t)$ is increasing in t. ∎

4. A linear lower bound for $B(t)$ when $D_A = D_B$. In this section we shall prove Theorem 2. We remind the reader that P without superscript stands for the the measure governing the Y-process when the initial $N_A(x, 0-)$ are i.i.d. mean μ_A Poisson variables and a finite number of B-particles are added at time 0. Throughout this section we assume that the A and B-particles perform random walks with the same distribution, that is

$$D_A = D_B. \tag{4.1}$$

Therefore, the paths $\pi_A(\cdot, \rho)$ and $\pi_B(\cdot, \rho)$, which were introduced in the beginning of Section 2, have the same distribution. From this one can show that the distribution of the positions of all the particles will remain the same if we do not use the paths $\pi_B(\cdot, \rho)$ at all, that is if we replace π_B in (2.2) by π_A. For the case with $D_A = D_B$ we shall use this different construction. We then have that the position at time s

of a particle ρ which starts in z is $z + \pi_A(s, \rho)$ for all s. However, the type of ρ will change from A to B at $\theta(\rho)$, the first instant when ρ coincides with a B-particle. These paths $\pi_A(\cdot, \rho)$ for different ρ are independent and so, as far as position of the particles is concerned, there is no interaction. Thus the system of particles which start out as A-particles (i.e., all particles but the finitely many initial B-particles) is the same as the system $\mathcal{P}^*$ introduced in Section 2 just before (2.17), as far as positions of particles are concerned. In agreement with Section 2 we write $N^*(x, t)$ for the number of particles at the space time-point (x, t) which started out as an A-particle (but whose type may have changed to B by time t). Of course, this assumes, as before, that $\mathcal{P}^*$ is coupled with the true system such that $N^*(x, 0) = N_A(x, 0-)$. In the present setup this means that the $N^*(x, 0)$, $x \in \mathbb{Z}^d$, are i.i.d., mean μ_A Poisson variables. The system $\mathcal{P}^*$ is then stationary in time for $t \geq 0$. It is convenient to extend the system $\mathcal{P}^*$ to a stationary system defined for all times $t \in \mathbb{R}$, including negative ones. For our system of non interacting random walkers this can easily be done by extending the path $t \mapsto \pi_A(t, \rho)$ for each particle ρ present at time zero to all t in such a way that $\{-\pi_A(-t, \rho)\}_{t \geq 0}$ has the same distribution as $\{\pi_A(t, \rho)\}_{t \geq 0}$, and in such a way that all paths $\{\pi_A(\cdot, \rho')\}, \{\pi_A(\cdot, \rho'')\}$ (with ρ', ρ'' varying over all particles present at time 0) are independent. We shall still use the notation $\mathcal{P}^*$ for the extended system. The configurations $\{N^*(x, t), x \in \mathbb{Z}^d\}$ are stationary in time in $\mathcal{P}^*$.

We first give a rough outline of the proof of Theorem 2. The simplest case is the one with d = 1 and only one B-particle at the origin at time 0. In this one-dimensional case, there is for each t a rightmost B-particle, at position $R(t)$ say, and a leftmost B-particle at position $-L(t)$. At time t all particles in $[-L(t), R(t)]$ are B-particles and all particles outside $[-L(t), R(t)]$ are A-particles. Basically we want to show that $\liminf_{t>0} R(t)/t > 0$ and similarly for $L(t)$. If there is exactly one particle at $R(t)$ at time t, then the rightmost B-particle moves left or right by one unit, each at rate $D_B/2$. However, if there is more than one particle at $R(t)$ at time t, then the rightmost particle moves one step to the right as soon as one of the particles at $R(t)$ makes a jump to the right, whereas the rightmost position moves a step to the left only when all particles at $R(t)$ move to the left. Thus, the rightmost B-particle has a drift to the right at all times when there is more than one particle at $R(t)$. When there are several particles (of either type) "close to" the rightmost B-particle, then there is a positive probability that in the next time unit several particles will coincide with the rightmost B-particle. This will still provide $R(\cdot)$ with an upwards drift. The only way for $R(t)/t$ to become small (with a non-negligible probability) is if the particle at $R(s)$ has for most $s \in [0, t]$ no particle nearby. We therefore want to show that the probability of this event goes to 0. It seems to be very difficult to use any special properties of the path $s \mapsto R(s)$, because of the complicated dependence between $R(s)$ and the position of particles relative to $R(s)$. One therefore proves the stronger statement that for any $\varepsilon > 0$ there exists a constant $K_1 > 0$ such that with probability $1 - \varepsilon$ *any* "decent" path $s \mapsto \pi(s), 0 \leq s \leq t$, has several particles in $[\pi(s) - K_1, \pi(s) + K_1]$ for s in a subset of $[0, t]$ of measure at least $(1 - \varepsilon)t$. Here, a decent path is a path with at most $K_2 t$ jumps during $[0, t]$ for a K_2 which will be specified in the proof.

The situation is somewhat more complicated in higher dimensions, but for a given $x \in \mathbb{Z}^d$ one can still construct a path $t \mapsto \lambda(t,x)$ which has a drift towards x at every time t when there is more than one particle at $\lambda(t,x)$. Roughly speaking, one does this by letting λ jump with the first particle which jumps away from present position of λ only if such a jump takes λ closer to x. (The lines right after the proof of Proposition 9 give a somewhat more technical description of this drift.) As in dimension 1 this reduces the problem to showing that for any $\varepsilon > 0$

$$\begin{aligned}&\text{any decent path } s \mapsto \pi(s), 0 \le s \le t, \text{ has several particles in the cube}\\ &\pi(s) + \mathcal{C}(K_3) \text{ for } s \text{ in a subset of } [0,t] \text{ of measure at least } (1-\varepsilon)t.\end{aligned} \tag{4.2}$$

(4.2) basically says that there are no paths in space-time which spend much time in regions with a density of A-particles which is much too low. One therefore wants a large deviation estimate for the density of the A-particles in the downwards direction (along paths in space-time). This is a counterpart to Theorem 2 of [KS] in which we proved a large deviation estimate of this kind for deviations of the density in the upward direction. It turns out that one can more or less follow the lines of that proof for (4.2) as well. In fact the proof for (4.2) is considerably simpler than the one used in [KS]. Note that (4.2) is a statement about the system of A-particles only; we are no longer concerned with the B-particles.

To explain how (4.2) is proven we need some more notation. This notation is more or less the same as in [KS], but some quantities (notably γ_r and Φ_r) have altered definitions. As in the previous sections $N_A(x,t)$ and $N_B(x,t)$ denote the number of A-particles and B-particles at (x,t). A *path* $\pi = (x_0, x_1, \dots, x_m)$ is now a sequence of vertices of $\mathbb{Z}^d$ with $x_{j+1} - x_j = \pm e_i, 1 \le i \le d$. We regard the x_j as the succesive positions of a space-time path $\widehat{\pi}$. There are many space-time paths which traverse the same positions. A *space-time path* $\widehat{\pi}$ is specified by giving its successive positions x_i and jumptimes s_i. For $s_1 < s_2 \dots$ we shall sometimes denote the path which jumps to x_i at time s_i by $\widehat{\pi}(\{s_i, x_i\})$. We make the convention that $s_0 = 0$, and unless stated otherwise, $x_0 = \mathbf{0}$. In addition we are here only discussing space-time paths over the time interval $[0,t]$, so we tacitly take $s_m \le t$. $\widehat{\pi}(\{s_i, x_i\})$ is then the path which is at position x_i during $[s_i, s_{i+1})$ for $0 \le i < m$, and at position x_m during $[s_m, t]$. If it is important that the path has exactly m jumptimes, then we shall write $\widehat{\pi}(\{s_i, x_i\}_{i \le m})$. Throughout this proof we shall only consider paths which are contained in

$$\mathcal{C}(t \log t) = [t \log t, t \log t]^d.$$

We shall be particularly interested in the following class of paths with exactly ℓ jumps:

$$\Xi(\ell, t) = \{\widehat{\pi}(\{s_i, x_i\}_{0 \le i \le \ell}) \text{ with } 0 < s_1 < \dots < s_\ell < t \text{ and } x_i \in \mathcal{C}(t \log t)\}. \tag{4.3}$$

Next, for some large integer C_0 and $r = 1, 2, \dots$, we take

$$\Delta_r = C_0^{6r}, \tag{4.4}$$

$$\mathcal{B}_r(\mathbf{i}, k) := \prod_{s=1}^{d} [i(s)\Delta_r, (i(s)+1)\Delta_r) \times [k\Delta_r, (k+1)\Delta_r), \tag{4.5}$$

$$\widetilde{\mathcal{B}}_r(\mathbf{i}, k) := \prod_{s=1}^{d} [(i(s)-3)\Delta_r, (i(s)+4)\Delta_r) \times [(k-1)\Delta_r, (k+1)\Delta_r). \tag{4.6}$$

For $x = (x(1), \dots, x(d)) \in \mathbb{Z}^d$ we further write

$$\mathcal{Q}_r(x) = \prod_{s=1}^{d} [x(s), x(s) + C_0^r). \tag{4.7}$$

Note that the edge size of the cube $\mathcal{Q}_r$ is only C_0^r and not Δ_r. We also introduce the random variables

$$U_r(x, v) = \sum_{y \in \mathcal{Q}_r(x)} N^*(y, v) = \sum_{\substack{y: x(s) \le y(s) < x(s) + C_0^r \\ 1 \le s \le d}} N^*(y, v). \tag{4.8}$$

(We shall need this only for integer times v.) Note that the so-called r-blocks $\mathcal{B}_r(\mathbf{i}, k)$, $\mathbf{i} \in \mathbb{Z}^d, k \in \mathbb{Z}$, form a partition of space-time, $\mathbb{Z}^d \times \mathbb{R}$. The r-block $\mathcal{B}_r(\mathbf{i}, k)$ is called *bad* if

$$U_r(x, v) < \gamma_r \mu_A C_0^{dr} \text{ for some } (x, v) \text{ with integer } v \text{ for which}$$
$$\mathcal{Q}_r(x) \times \{v\} \text{ is contained in } \widetilde{\mathcal{B}}_r(\mathbf{i}, k). \tag{4.9}$$

The γ_r in this definition are given by (4.17) below. For the time being the only important property is that

$$0 < \gamma_0 < \gamma_r < \gamma_\infty \le \frac{1}{2}, \; r \ge 0. \tag{4.10}$$

Roughly speaking, the bad blocks are blocks in which the density of A-particles is lower than half the expected density on some subcube of specified size. Finally we define for any space-time path $\widehat{\pi}$

$$\phi_r(\widehat{\pi}) = \text{number of bad } r\text{-blocks which intersect the space-time path } \widehat{\pi} \tag{4.11}$$

and

$$\Phi_r(\ell) = \sup_{\widehat{\pi} \in \,\Xi(\ell, t)} \phi_r(\widehat{\pi}). \tag{4.12}$$

We suppress the dependence on t of Φ_r and related quantities in our notation. This is unlikely to cause confusion.

The principal estimate is now the recursive relation

$$\Phi_r(\ell) \le C_0^{6(d+1)} \Phi_{r+1}(\ell) + C_0^{6(d+1)} \Psi_{r+1}(\ell) \tag{4.13}$$

for some random variable $\Psi_{r+1}(\ell)$ which is small in the probabilistic sense made precise in Lemma 7. (4.13) and Lemma 7 are based on an estimate for

$$P\{\mathcal{B}_{r+1}(\mathbf{i},k) \text{ is good, but contains a bad } r\text{-block}\},$$

plus an argument from percolation. A simple estimate for the Poisson distribution shows further that for any choice of K there exists a $K_4 = K_4(K, d, \mu_A)$ such that if $R = R(t)$ satisfies

$$C_0^R \geq [K_4 \log t]^{1/d} > C_0^{R-1},$$

then

$$P\{\Phi_r(\ell) > 0 \text{ for any } r \geq R \text{ and any } \ell \geq 0\} \leq \frac{1}{t^K} \tag{4.14}$$

for large t. Once we have (4.13)-(4.14) and the probability estimate of Lemma 7 for Ψ_{r+1}, it follows easily that also

$$P\{\Phi_r(\ell) \geq \varepsilon_0 C_0^{-6r}(t+\ell) \text{ for some } r \geq r_0, \ell \geq 0\} \leq \frac{2}{t^K}. \tag{4.15}$$

for large t (see Proposition 9). This is essentially the desired (4.2). Indeed, except for showing that we only need to consider $\ell \leq K_4 t$, (4.15) for some large r shows that the relevant space-time paths intersect few bad r-blocks.

We now fill in the details. The constants C_0 and γ_r already mentioned above are chosen as follows: $\gamma_0 > 0$ is a constant which satisfies

$$0 < \gamma_0 \prod_{j=1}^{\infty} [1 - 2^{-j/4}]^{-1} \leq \frac{1}{2}. \tag{4.16}$$

We take

$$\gamma_1 = \gamma_0, \quad \gamma_{r+1} = \gamma_0 \prod_{j=1}^{r} \Big[1 - \frac{1}{C_0^{j/4}}\Big]^{-1}, \ r > 0. \tag{4.17}$$

Then (4.16) together with $C_0 \geq 2$, implies (4.10). Further, C_0 is an integer which is so large that for all $r \geq 1$,

$$C_0^{-r/2} - \Big(1 - \frac{C_4 (r \log C_0)^d}{C_0^r}\Big)(1 - e^{-C_0^{-r/2}})\big[1 - C_0^{-r/4}\big]^{-1} \leq -\frac{1}{2} C_0^{-3r/4}, \tag{4.18}$$

as well as

$$3^{d+1} C_0^{6(d+1)(r+1)} \exp\big[-\frac{1}{2}\gamma_0 \mu_A C_0^{(d-\frac{3}{4})r}\big] \leq 1, \ r \geq 1. \tag{4.19}$$

Here C_4 is the constant of Lemma 6 below. Since C_4 will not depend on C_0, we can indeed fulfill (4.18) and (4.19) by taking C_0 large. The $\Delta_r, \mathcal{B}_r, \widetilde{\mathcal{B}}_r$ and U_r are defined in (4.4)-(4.8). In addition we need

$$\widetilde{\mathcal{B}}_r^+(\mathbf{i},k) := \prod_{s=1}^{d} \big[(i(s)-1)\Delta_r, (i(s)+2)\Delta_r\big) \times \big[k\Delta_r - \Delta_{r-1}, (k+1)\Delta_r\big),$$

$$V_r(\mathbf{i}) = \prod_{s=1}^d \big[(i(s)-3)\Delta_r, (i(s)+4)\Delta_r\big),$$

and the *pedestal* of $\mathcal{B}_r(\mathbf{i},k)$ which is defined as

$$\mathcal{V}_r(\mathbf{i},k) = V_r(\mathbf{i}) \times \{(k-1)\Delta_r\} = \prod_{s=1}^d \big[(i(s)-3)\Delta_r, (i(s)+4)\Delta_r\big) \times \{(k-1)\Delta_r\}.$$

In analogy with the definition of a bad block we call the pedestal $\mathcal{V}_r(\mathbf{i},k)$ bad if

$$U_r(x,(k-1)\Delta_r) < \gamma_r \mu_A C_0^{dr} \text{ for some } x \text{ with } \mathcal{Q}_r(x) \subset V_r(\mathbf{i}). \tag{4.20}$$

A block or pedestal is called *good* if it is not bad. Note that in contrast to [KS], the good blocks and pedestals have $U(x,v)$ large.

For a space-time path $\widehat{\pi}(\{s_i,x_i\})$ we define in addition to the ϕ_r, Φ_r of (4.11), (4.12)

$$\psi_{r+1}(\widehat{\pi}) = \text{number of } (r+1)\text{-blocks which intersect the space-time path } \widehat{\pi} \text{ and which have a good pedestal but contain a bad } r\text{-block} \tag{4.21}$$

and

$$\Psi_r(\ell) = \sup_{\widehat{\pi} \in \, \Xi(\ell,t)} \psi_r(\widehat{\pi}). \tag{4.22}$$

Exactly as in Lemma 8 of [KS] we now have for any $\widehat{\pi} \in \Xi(\ell,t)$

$$\phi_r(\widehat{\pi}) \le C_0^{6(d+1)} \Phi_{r+1}(\ell) + C_0^{6(d+1)} \psi_{r+1}(\widehat{\pi}) \tag{4.23}$$

and

$$\Phi_r(\ell) \le C_0^{6(d+1)} \Phi_{r+1}(\ell) + C_0^{6(d+1)} \Psi_{r+1}(\ell). \tag{4.24}$$

Now choose some large constant K and take $R = R(t)$ to be the integer for which

$$C_0^R \ge [K_4 \log t]^{1/d} > C_0^{R-1}. \tag{4.25}$$

(This differs slightly from [KS], which had a one instead of the K_4 here.) As in [KS], Lemmas 5 and 9, simple Poisson distribution estimates now show that we can take $K_4 = K_4(K,d)$ so large that

$$\begin{aligned} &P\{\Phi_r(\ell) > 0 \text{ for any } r \ge R \text{ and any } \ell \ge 0\} \\ &\le \sum_{r \ge R} P\{U_r(x,v) < \frac{1}{2}\mu_A C_0^{dr} \text{ for some } (x,v) \text{ with integer } v \in [-\Delta_r, t+\Delta_r) \\ &\qquad \text{for which } \mathcal{Q}_r(x) \text{ intersects } \mathcal{C}(t\log t + 3\Delta_r) \\ &\le t^{-K} \text{ for all large } t. \end{aligned} \tag{4.26}$$

Note that we used $\gamma_r \le 1/2$. Note also that the required value of K_4 depends on K, d and μ_A only.

We shall also need the following analogue of Lemma 6 in [KS] (note that this time the inequality goes in the opposite direction from (5.22) in [KS]). $\{S_u\}_{u \ge 0}$ is short for what we formerly denoted by $\{S_u^A\}_{u\ge 0}$, that is, a copy of the random walk performed by the A-particles (starting at $\mathbf{0}$).

Lemma 6. *There exist a constant $C_4 = C_4(d, D_A)$, which is independent of C_0, such that if $\mathcal{V}_{r+1}(\mathbf{i}, k)$ is good, and u an integer with $\Delta_{r+1} - \Delta_r \le u \le 2\Delta_{r+1}$, then for $r \ge 1$, and*

$$y \in \prod_{s=1}^{d} \big[(i(s)-1)\Delta_{r+1}, (i(s)+2)\Delta_{r+1}\big), \tag{4.27}$$

it holds

$$\sum_{z \in V_{r+1}(\mathbf{i})} N^*(z, (k-1)\Delta_{r+1}) P\big\{z + S_u \in \mathcal{Q}_r(y)\big\}$$
$$\ge \gamma_{r+1}\mu_A C_0^{dr}\Big[1 - \frac{C_4(r \log C_0)^d}{C_0^r}\Big]. \tag{4.28}$$

Proof. Let $r \ge 1$ be fixed. In addition to the blocks $\prod_{s=1}^d [i(s)\Delta_{r+1}, (i(s)+1)\Delta_{r+1})$ which have edge length $\Delta_{r+1} = C_0^{6(r+1)}$, we also need the blocks

$$\mathcal{M}(\mathbf{j}) := \prod_{s=1}^{d} [j(s)C_0^{r+1}, (j(s)+1)C_0^{r+1}).$$

In our previous notation $\mathcal{M}(\mathbf{j}) = \mathcal{Q}_{r+1}(x)$ with $x(s) = j(s)C_0^{r+1}$. These blocks have edge length only C_0^{r+1}, and the set $V_{r+1}(\mathbf{i})$ is a disjoint union of $7^d C_0^{5d(r+1)}$ of these smaller blocks. Let $\Lambda = \Lambda(\mathbf{i}, r+1)$ be the set of $\mathbf{j} \in \mathbb{Z}^d$ with

$$\mathcal{M}(\mathbf{j}) \subset V_{r+1}(\mathbf{i}).$$

Also, for each $\mathbf{j} \in \Lambda$ let $z_{\mathbf{j}} \in \mathcal{M}(\mathbf{j})$ be such that

$$P\big\{z_{\mathbf{j}} + S_u \in \mathcal{Q}_r(y)\big\} = \min_{z \in \mathcal{M}(\mathbf{j})} P\big\{z + S_u \in \mathcal{Q}_r(y)\big\}.$$

Then the left hand side of (4.28) equals

$$\sum_{\mathbf{j} \in \Lambda} \sum_{z \in \mathcal{M}(\mathbf{j})} N^*(z, (k-1)\Delta_{r+1}) P\big\{z + S_u \in \mathcal{Q}_r(y)\big\}$$
$$\ge \sum_{\mathbf{j} \in \Lambda} \sum_{z \in \mathcal{M}(\mathbf{j})} N^*(z, (k-1)\Delta_{r+1}) P\big\{z_{\mathbf{j}} + S_u \in \mathcal{Q}_r(y)\big\}. \tag{4.29}$$

Since $\mathcal{V}_{r+1}(\mathbf{i}, k)$ is assumed to be good, we have

$$\sum_{z \in \mathcal{M}(\mathbf{j})} N^*(z, (k-1)\Delta_{r+1}) = U_{r+1}(\mathbf{j}C_0^{r+1}, (k-1)\Delta_{r+1})$$
$$\ge \gamma_{r+1}\mu_A C_0^{d(r+1)} = \sum_{z \in \mathcal{M}(\mathbf{j})} \gamma_{r+1}\mu_A.$$

We can therefore continue (4.29) to obtain that the left hand side of (4.28) is at least

$$\begin{aligned}
&\sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})}\gamma_{r+1}\mu_A P\big\{z_{\mathbf{j}}+S_u\in\mathcal{Q}_r(y)\big\}\\
&\geq \sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})}\gamma_{r+1}\mu_A P\big\{z+S_u\in\mathcal{Q}_r(y)\big\}\\
&\qquad -\sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})}\gamma_{r+1}\mu_A\Big|P\big\{z_{\mathbf{j}}+S_u\in\mathcal{Q}_r(y)\big\}-P\big\{z+S_u\in\mathcal{Q}_r(y)\big\}\Big|. \qquad (4.30)
\end{aligned}$$

Now, by virtue of (4.27) the first multiple sum in the right hand side of (4.30) is at least

$$\begin{aligned}
&\sum_{z:z-y\in[-2\Delta_{r+1},2\Delta_{r+1})^d}\gamma_{r+1}\mu_A\sum_{w\in\mathcal{Q}_r(y-z)}P\{S_u=w\}\\
&\geq \sum_{w\in[-\Delta_{r+1},\Delta_{r+1})^d}P\{S_u=w\}\sum_{z\in\mathcal{Q}_r(y-w)}\gamma_{r+1}\mu_A\\
&= \sum_{w\in[-\Delta_{r+1},\Delta_{r+1})^d}P\{S_u=w\}\gamma_{r+1}\mu_A C_0^{dr}\\
&=\gamma_{r+1}\mu_A C_0^{dr}\big[1-P\{S_u\notin[-\Delta_{r+1},\Delta_{r+1})^d\}\big]\\
&\geq\gamma_{r+1}\mu_A C_0^{dr}\big[1-K_5\exp[-K_6\Delta_{r+1}]\big] \qquad (4.31)
\end{aligned}$$

for some constants K_5, K_6, depending on d, D_A only. In the last inequality we used simple large deviation estimates for S_u (see for instance (2.40) in [KS]) and the fact that $u\leq 2\Delta_{r+1}$.

On the other hand, we have for any $z\in\mathcal{M}(\mathbf{j})$ that

$$\begin{aligned}
&\Big|P\big\{z_{\mathbf{j}}+S_u\in\mathcal{Q}_r(y)\big\}-P\big\{z+S_u\in\mathcal{Q}_r(y)\big\}\Big|\\
&\leq\sum_{w\in\mathcal{Q}_r(y)}\big|P\{z_{\mathbf{j}}+S_u=w\}-P\{z+S_u=w\}\big|\\
&\leq\sum_{v\in\mathcal{Q}_r(y-z)}\sup_{w:\|w-v\|\leq C_0^{r+1}}\big|P\{S_u=v\}-P\{S_u=w\}\big|.
\end{aligned}$$

It follows that the second multiple sum in the right hand side of (4.30) is bounded in absolute value by

$$\begin{aligned}
&\sum_z\gamma_{r+1}\mu_A\sum_{v\in\mathcal{Q}_r(y-z)}\sup_{w:\|w-v\|\leq C_0^{r+1}}\big|P\{S_u=v\}-P\{S_u=w\}\big|\\
&\leq\gamma_{r+1}\mu_A\sum_{v\in\mathbb{Z}^d}\sum_{z\in\mathcal{Q}_r(y-v)}\sup_{w:\|w-v\|\leq C_0^{r+1}}\big|P\{S_u=v\}-P\{S_u=w\}\big|\\
&=\gamma_{r+1}\mu_A C_0^{dr}\sum_{v\in\mathbb{Z}^d}\sup_{w:\|w-v\|\leq C_0^{r+1}}\big|P\{S_u=v\}-P\{S_u=w\}\big|.
\end{aligned}$$

The right hand side here has been estimated in (6.37) and in (5.26) and the following lines in [KS]. The result is that the right hand side here is bounded by

$$K_7 \gamma_{r+1} \mu_A C_0^{dr-2r-2} [r \log C_0]^d$$

for some constant K_7 which does not depend on C_0. Combining this with the estimates (4.30) and (4.31) we obtain (4.28). ∎

We define some σ-fields analogous to [KS] (but with some differences):

$$\begin{aligned}\mathcal{H}_{r+1}(\mathbf{i},k) := \;&\sigma\text{-field generated by the paths of all particles through}\\ &\text{time } (k-1)\Delta_{r+1} \text{ and the paths through time}\\ &(k+1)\Delta_{r+1}-1 \text{ of the particles which are outside}\\ &V_{r+1}(\mathbf{i}) \text{ at time } (k-1)\Delta_{r+1},\end{aligned} \tag{4.32}$$

$$\mathcal{K}_{r+1} := \sigma\text{-field generated by } \{N^*(x,(k-1)\Delta_{r+1}) : x \in V_{r+1}(\mathbf{i})\}.$$

Note that

$$\mathcal{K}_{r+1} \subset \mathcal{H}_{r+1}(\mathbf{i},k),$$

because if one knows all paths through time $(k-1)\Delta_{r+1}$, then one also knows how many particles there are at each x at time $(k-1)\Delta_{r+1}$. In other words, all $N^*(x,(k-1)\Delta_{r+1}), x \in \mathbb{Z}^d$, are $\mathcal{H}_{r+1}(\mathbf{i},k)$-measurable.

We also need certain events $\mathcal{A}(\mathbf{i},k)$ which are somewhat larger than the event $\{\mathcal{B}_{r+1}(\mathbf{i},k)$ contains some bad $\mathcal{B}_r(\mathbf{j},q)\}$. To define these events $\mathcal{A}$ we note that any space-time point (y,v) belongs to a unique $(r+1)$-block $\mathcal{B}_{r+1}(\mathbf{i},k)$. For $(y,v) \in \mathcal{B}_{r+1}(\mathbf{i},k)$ we define

$$\begin{aligned}W_r(y,v) = \;&\text{number of particles in the system } \mathcal{P}^* \text{ in } \mathcal{Q}_r(y) \times \{v\}\\ &\text{which were in } V_r(\mathbf{i}) \text{ at time } (k-1)\Delta_{r+1}.\end{aligned}$$

A block $\mathcal{B}_r(\mathbf{j},q) \subset \mathcal{B}_{r+1}(\mathbf{i},k)$ will be called *inferior* if $W_r(y,v) < \gamma_r \mu_A C_0^{dr}$ for some (y,v) for which v is an integer and $\mathcal{Q}_r(y) \times \{v\}$ is contained in $\widetilde{\mathcal{B}}_r(\mathbf{j},q)$.

It is apparent from the definitions that

$$W_r(y,v) \le U_r(y,v), \tag{4.33}$$

since we count only particles which passed through $\mathcal{V}_{r+1}(\mathbf{i},k)$ in $W_r(y,v)$, whereas $U_r(y,v)$ also counts particles which do not satisfy this requirement. It follows from this that a bad block is also inferior. Finally, we define the event

$$\mathcal{A}(\mathbf{i},k) = \mathcal{A}(\mathbf{i},k,r) = \{\mathcal{B}_{r+1}(\mathbf{i},k) \text{ contains some inferior } r\text{-block } \mathcal{B}_r(\mathbf{j},q).$$

One now has the following analogue of Lemma 7 and part of the proof of Lemma 8 in [KS].

Lemma 7. *Let*

$$\rho_{r+1} = 3^{d+1} C_0^{6(d+1)(r+1)} \exp\big[-\frac{1}{2}\gamma_r \mu_A C_0^{(d-\frac{3}{4})r}\big],\ r \ge 1. \tag{4.34}$$

Then for $r \ge 1$, on the event $\{\mathcal{V}_{r+1}(\mathbf{i},k)$ is good$\}$,

$$P\{\mathcal{A}(\mathbf{i},k)\big|\mathcal{H}_{r+1}(\mathbf{i},k)\} = P\{\mathcal{A}(\mathbf{i},k)\big|\mathcal{K}_{r+1}(\mathbf{i},k)\} \le \rho_{r+1}. \tag{4.35}$$

Moreover, for fixed $a(s) \in \{0,1,\dots,11\}$ and $b = 0$ or 1, the collection of pairs $(\mathbf{i},k), i(s) \equiv a(s) \bmod 12, 1 \le s \le d, k \equiv b \bmod 2$, for which $\mathcal{V}_{r+1}(\mathbf{i},k)$ is good, but $\mathcal{A}(\mathbf{i},k)$ occurs, is stochastically smaller than an independent percolation system in which each site $(\mathbf{i},k), i(s) \equiv a(s) \bmod 12, 1 \le s \le d, k \equiv b \bmod 2$, is open with probability ρ_{r+1}.

Proof. By the Markov property of the particle system $\mathcal{P}^*$, the conditional distribution of the particles during $[(k-1)\Delta_{r+1}, \infty)$, given the behavior of all particles during $(-\infty, (k-1)\Delta_{r+1}]$ is the same as the conditional distribution given the positions of all particles at time $(k-1)\Delta_{r+1}$. Moreover, given the positions of the particles at time $(k-1)\Delta_{r+1}$, the future paths of all particles are conditionally independent. In particular, given the particles at time $(k-1)\Delta_{r+1}$, the paths after time $(k-1)\Delta_{r+1}$ of the particles in $\mathcal{V}_{r+1}(i,k)$, are conditionally independent of the future paths of all particles outside $\mathcal{V}_{r+1}(\mathbf{i},k)$ at time $(k-1)\Delta_{r+1}$. By definition the event $\mathcal{A}(\mathbf{i},k)$ depends only on the particles in $\mathcal{V}_{r+1}(\mathbf{i},k)$ at time $(k-1)\Delta_{r+1}$ and the increments of their paths after time $(k-1)\Delta_{r+1}$. It follows from these comments that $P\{\mathcal{A}(\mathbf{i},k)\big|\mathcal{H}_{r+1}(\mathbf{i},k)\}$ is a function of the particles in $\mathcal{V}_{r+1}(\mathbf{i},k)$ only. In fact, since $\mathcal{A}(\mathbf{i},k)$ depends on particle counts only, $P\{\mathcal{A}(\mathbf{i},k)\big|\mathcal{H}_{r+1}(\mathbf{i},k)\}$ depends only on the $N^*(x,v)$ with $(x,v) \in \mathcal{V}_{r+1}(\mathbf{i},k)$ (see the explicit computation in the next paragraph). Thus the left hand side of (4.35) is $\mathcal{K}_{r+1}(\mathbf{i},k)$-measurable, and equals the middle member of (4.35).

We next prove the inequality in (4.35). If $\mathcal{A}(\mathbf{i},k)$ occurs, then $W_r(y,v) < \gamma_r \mu_A C_0^{dr}$ for some integer v and

$$\begin{aligned}(y,v) &\in \bigcup_{B_r(\mathbf{j},q) \subset B_{r+1}(\mathbf{i},k)} \widetilde{B}_r(\mathbf{j},q) \\ &\subset \prod \big[i(s)\Delta_{r+1} - 3\Delta_r, (i(s)+1)\Delta_{r+1} + 3\Delta_r\big) \times [k\Delta_{r+1} - \Delta_r, (k+1)\Delta_{r+1}) \\ &\subset \prod \big[(i(s)-1)\Delta_{r+1}, (i(s)+2)\Delta_{r+1} - \Delta_r\big) \times [k\Delta_{r+1} - \Delta_r, (k+1)\Delta_{r+1}).\end{aligned} \tag{4.36}$$

Now consider any (y,v) satisfying (4.36) and let the particles in $\mathcal{V}_{r+1}(\mathbf{i},k)$ be given such that $\mathcal{V}_{r+1}(\mathbf{i},k)$ is good. Conditionally on this, the distribution of $W_r(y,v)$ is the distribution of

$$\sum_{z \in V_{r+1}(\mathbf{i})} \sum_{r=1}^{N^*(z,(k-1)\Delta_{r+1})} I\big[z + S_u^{z,r} \in \mathcal{Q}_r(y)\big],$$

where the $\{S^{z,j}\}$ are independent copies of the random walk $\{S\}$ and $u = v - (k-1)\Delta_{r+1} \in [\Delta_{r+1} - \Delta_r, 2\Delta_{r+1}]$ (see the proof of Lemma 7, and in particular the lines following (5.37) in [KS]). Therefore, $P\{W_r(y,v) < \gamma_r \mu_A C_0^{dr}\}$ is the probability of fewer than $\gamma_r \mu_A C_0^{dr}$ successes in

$$\sum_{z \in V_{r+1}(\mathbf{i})} N^*(z, (k-1)\Delta_{r+1})$$

trials, $N^*(z, (k-1)\Delta_{r+1})$ of which have success probability

$$p(y-z, u) := P\{z + S_u \in \mathcal{Q}_r(y)\}.$$

Very much as in (5.38), (5.39) of [KS] we therefore have for $\theta \geq 0$,

$$\begin{aligned} &E\{\exp[-\theta W_r(y,v)] \big| \mathcal{K}_{r+1}\} \\ &= \prod_{z \in V_{r+1}(\mathbf{i})} \big[1 - p(y-z,u) + p(y-z,u)e^{-\theta}\big]^{N^*(z,(k-1)\Delta_{r+1})} \\ &\leq \exp\Big[-\sum_{z \in V_{r+1}(\mathbf{i})} N^*(z,(k-1)\Delta_{r+1}) p(y-z,u)(1-e^{-\theta})\Big]. \end{aligned}$$

For $\theta = C_0^{-r/2}$ this gives, by virtue of (4.17),(4.18) and Lemma 6, that on the event $\{\mathcal{V}_{r+1}(\mathbf{i},k) \text{ is good}\}$ and for (y,v) satisfying (4.36),

$$\begin{aligned} &P\{W_r(y,v) < \gamma_r \mu_A C_0^{dr} \big| \mathcal{K}_{r+1}\} \\ &\leq \exp\Big[\theta \gamma_r \mu_A C_0^{dr} - \sum_{z \in V_{r+1}(\mathbf{i})} N^*(z,(k-1)\Delta_{r+1}) p(y-z,u)(1-e^{-\theta})\Big] \\ &\leq \exp\big[-\frac{1}{2}\gamma_r \mu_A C_0^{(d-\frac{3}{4})r}\big]. \end{aligned} \tag{4.37}$$

(4.35) now follows from the fact that $P\{\mathcal{A}(\mathbf{i},k)\}$ is bounded by the sum over (y,v) with integral v and satisfying (4.36).

The last statement of the lemma concerning the stochastic ordering between the collection of pairs $(\mathbf{i},k)$ for which $\mathcal{V}_{r+1}(\mathbf{i},k)$ is good and $\mathcal{A}(\mathbf{i},k)$ occurs, and an independent percolation system, now follows in exactly the same way as in the proof of (5.43) in [KS]. ∎

The next lemma is basically a copy of parts of Lemma 8 and Lemma 11 in [KS]. Note that now $R = R(t)$ is defined in (4.25).

Lemma 8. *The inequalities* (4.23) *and* (4.24) *hold. Moreover, there exist some constants* $C_5 = C_5(d,\mu_A)$, $\kappa_0 = \kappa(d,\mu_A)$ *and* $t_0 = t_0(d,\mu_A)$ *(independent of* r,ℓ*), such that for* $1 \leq r \leq R(t)-1, \kappa \geq \kappa_0, t \geq t_0$*, and any* $\ell \geq 0$

$$\begin{aligned} &P\Big\{\Psi_{r+1}(\ell) \geq \frac{\kappa(t+\ell)}{\Delta_{r+1}}[\rho_{r+1}]^{1/(d+1)}\Big\} \\ &\qquad \leq \exp\Big[-(t+\ell)C_5\kappa \exp\big[-\frac{1}{2(d+1)}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big]\Big]. \end{aligned} \tag{4.38}$$

Proof. We already observed that (4.23) and (4.24) hold, for the same reasons as in Lemma 8 of [KS].

The inequality (4.38) follows by a percolation argument which is given in the proof of (6.28) and Lemma 8 of [KS]; see also proof of Theorem 9 in [L]. This time we take an integer ν such that

$$[\rho_{r+1}]^{-1/(d+1)} \le \nu \le 2[\rho_{r+1}]^{-1/(d+1)} \tag{4.39}$$

and define

$$\mathcal{D}(\mathbf{m}, q) = \prod_{s=1}^{d} [\nu m(s)\Delta_{r+1}, \nu(m(s)+1)\Delta_{r+1}) \times [q\nu\Delta_{r+1}, (q+1)\nu\Delta_{r+1}). \tag{4.40}$$

$\mathbf{m}$ here is short for $(m(1), \dots, m(d))$; the $\mathbf{m}$ and ν here have nothing to do with the m_i and ν in the proof of Proposition 5. Note that $\rho_{r+1} \le 1$ by (4.19) and (4.10), so that (4.39) can be satisfied. Each $\mathcal{D}(\mathbf{m}, q)$ is the disjoint union of ν^{d+1} $(r+1)$-blocks. Moreover, as shown in (6.30) of [KS], for $\ell \ge 0$ at most

$$\lambda(\ell) := 3^d \Big(\frac{t+\ell}{\nu\Delta_{r+1}} + 2 \Big) \tag{4.41}$$

blocks $\mathcal{D}(\mathbf{m}, q)$ can intersect a space-time path $\widehat{\pi} \in \Xi(\ell, t)$ (with jumptimes $s_1 < \cdots < s_\ell < t$ and positions $x_1, \dots, x_\ell$). Now fix $a(1), \dots, a(d) \in \{0, \dots, 11\}, b \in \{0, 1\}$ and define for any space-time path $\widehat{\pi}$

$$\begin{aligned} \psi_{r+1}(\widehat{\pi}, \mathbf{a}, k) = {} & \text{number of } (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i}, k) \text{ with } i(s) \equiv a(s) \bmod 12, \\ & k \equiv b \bmod 2, \text{ which intersect the space-time path } \widehat{\pi} \text{ and} \\ & \text{which have a good pedestal but contain a bad } r\text{-block.} \end{aligned} \tag{4.42}$$

Define further

$$\Psi_{r+1}(\ell, \mathbf{a}, b) = \sup_{\widehat{\pi} \in \Xi(\ell, t)} \psi_{r+1}(\widehat{\pi}, \mathbf{a}, k).$$

Then

$$\Psi_{r+1}(\ell) \le \sum_{(\mathbf{a}, b)} \Psi_{r+1}(\ell, \mathbf{a}, b). \tag{4.43}$$

As in [KS], let $Z(\mathbf{i}, k)$ be independent random variables with

$$P\{Z(\mathbf{i}, k) = 1\} = 1 - P\{Z(\mathbf{i}, k) = 0\} = \rho_{r+1}.$$

Then, as in (6.31) of [KS],

$$P\Big\{\Psi_{r+1}(\ell,\mathbf{a},b) \ge 2^{-1}(12)^{-d}\frac{\kappa(t+\ell)}{\Delta_{r+1}}[\rho_{r+1}]^{1/(d+1)}\Big\}$$
$$\le \sum_{\mathcal{D}(\mathbf{m}_0,0),\dots,\mathcal{D}(\mathbf{m}_{\lambda-1},\lambda-1)} P\{\bigcup_{q=0}^{\lambda-1}\mathcal{D}(\mathbf{m}_q,q) \text{ contains at least } 2^{-1}(12)^{-d}\frac{\kappa(t+\ell)}{\Delta_{r+1}}[\rho_{r+1}]^{1/(d+1)} \; (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i},k) \text{ with } Z(\mathbf{i},k)=1, \text{ and } i(s)\equiv a(s) \bmod 12, k\equiv b \bmod 2\Big\}. \tag{4.44}$$

Here $(\mathcal{D}(\mathbf{m}_0,0),\dots,\mathcal{D}(\mathbf{m}_{\lambda-1},\lambda-1))$ runs over the possible collections of blocks $\mathcal{D}$ which intersect a space-time path $\widehat{\pi}\in\Xi(\ell,t)$. For some constant K_8 which depends on d only, there are at most

$$[2t\log t+1]^d\exp[K_8\lambda] \tag{4.45}$$

collections of this form. If we fix such a collection $\mathcal{D}(\mathbf{m}_0,0),\dots,\mathcal{D}(\mathbf{m}_{\lambda-1},\lambda-1)$, then the probability that

$$\bigcup_{q=0}^{\lambda-1}\mathcal{D}(\mathbf{m}_q,q)$$

contains at least $2^{-1}(12)^{-d}\kappa(t+\ell)\Delta_{r+1}^{-1}[\rho_{r+1}]^{1/(d+1)}$ $(r+1)$-blocks $\mathcal{B}_{(r+1)}(\mathbf{i},k)$ with $Z(\mathbf{i},k)=1$ and $(\mathbf{i},k)\equiv(\mathbf{a},b)$, is bounded by

$$P\Big\{T\ge 2^{-1}(12)^{-d}\frac{\kappa(t+\ell)}{\Delta_{r+1}}[\rho_{r+1}]^{1/(d+1)}\Big\}, \tag{4.46}$$

where T has a binomial distribution corresponding to $\lambda\nu^{d+1}$ trials with success probability ρ_{r+1}. As in (6.33) or (5.52) in [KS] one obtains from Bernstein's inequality (together with (4.39),(4.41) and (4.45)) that this probability is at most

$$K_9\exp\big[-K_{10}\frac{\kappa(t+\ell)}{\Delta_{r+1}}[\rho_{r+1}]^{1/(d+1)}\big]$$

for $1\le r\le R(t)-1$, $\kappa\ge$ some κ_0, $t\ge$ some t_0, and constants K_9,K_{10}, depending on d and μ_A only. The inequality (4.38) now follows from (4.44), (4.45) and (4.43). ∎

Proposition 9. *For any choice of K and $\varepsilon_0>0$, there exist constants r_0,t_1 such that for all $t\ge t_1$,*

$$P\{\Phi_r(\ell)\ge\varepsilon_0 C_0^{-6r}(t+\ell) \text{ for some } r\ge r_0,\ell\ge 0\}\le\frac{2}{t^K}. \tag{4.47}$$

Proof. Consider a sample point for which

$$\Phi_r(\ell) = 0 \text{ for all } r \geq R(t) \text{ and } \ell \geq 0 \tag{4.48}$$

and for which

$$\Phi_r(\ell) \leq C_0^{6(d+1)}\Phi_{r+1}(\ell) + C_0^{6(d+1)}\frac{\kappa_0(t+\ell)}{\Delta_{r+1}}[\rho_{r+1}]^{1/(d+1)} \tag{4.49}$$

for all $t \geq t_0, 1 \leq r \leq R-1, \ell \geq 0$. For such a sample point one also has for $t \geq t_0, r_0 \leq r \leq R-1, \ell \geq 0$,

$$\begin{aligned}
\Phi_r(\ell) &\leq C_0^{6(d+1)}\frac{\kappa_0(t+\ell)}{\Delta_{r+1}}[\rho_{r+1}]^{1/(d+1)} + C_0^{6(d+1)}\Phi_{r+1}(\ell) \\
&\leq C_0^{6(d+1)}3\kappa_0(t+\ell)\exp\Big[-\frac{\gamma_0\mu_A}{2(d+1)}C_0^{(d-\frac{3}{4})r}\Big] + C_0^{6(d+1)}\Phi_{r+1}(\ell) \\
&\leq C_0^{6(d+1)}3\kappa_0(t+\ell)\exp\Big[-\frac{\gamma_0\mu_A}{2(d+1)}C_0^{(d-\frac{3}{4})r}\Big] \\
&\qquad + C_0^{12(d+1)}3\kappa_0(t+\ell)\exp\Big[-\frac{\gamma_0\mu_A}{2(d+1)}C_0^{(d-\frac{3}{4})(r+1)}\Big] + C_0^{12(d+1)}\Phi_{r+2}(\ell) \\
&\leq \cdots \leq \sum_{j=1}^{R-r} C_0^{6j(d+1)}3\kappa_0(t+\ell)\exp\Big[-\frac{\gamma_0\mu_A}{2(d+1)}C_0^{(d-\frac{3}{4})(r+j-1)}\Big] \\
&\qquad + C_0^{6(d+1)(R-r)}\Phi_R(\ell) \\
&\leq 6\kappa_0(t+\ell)C_0^{6(d+1)}\exp\Big[-\frac{\gamma_0\mu_A}{2(d+1)}C_0^{(d-\frac{3}{4})r}\Big] \\
&\leq \varepsilon_0 C_0^{-6r}(t+\ell),
\end{aligned} \tag{4.50}$$

provided r_0 is sufficiently large. The required value for r_0 is independent of t, ℓ.

By (4.26),(4.38), (4.24) and (4.25), the relations (4.48) and (4.49) hold outside a set of probability

$$t^{-K} + \sum_{r=1}^{R-1}\sum_{\ell\geq 0}\exp\Big[-(t+\ell)C_5\kappa_0\exp\big[-\frac{1}{2(d+1)}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big]\Big] \leq 2t^{-K},$$

provided $t \geq$ some $t_1 \geq t_0$. This proves the Proposition. ■

Proposition 9 is the main technical estimate for the proof of Theorem 2. We now start on this proof proper. The strategy will be to show that (with overwhelming probability) there exists a (random) path $u \mapsto \lambda(u, x)$ along which a B-particle moves with a "drift" towards a fixed point x, at least at the times u when there are at least two particles at $\lambda(u, x)$. Lemma 10 below expresses this more formally; at the times when $I_{\geq 2}(u) = 1$ (and hence $I_1(u) = 0$), $\|\lambda(u, x) - x\|_2$ has a drift $D_B\Gamma_{\geq 2}(u)$, which will be shown to be negative in (4.81). A few steps in the proof can be done simpler if $D_A = D_B$. Nevertheless we have not always used this hypothesis in the hope that we can use much of the proof later even when $D_A \neq D_B$. The principal step for which $D_A = D_B$ is crucial is the proof of Proposition 9.

We now give the details. For $x \in \mathbb{Z}^d$ construct a path $\lambda(\cdot) = \lambda(\cdot, x) \in \mathbb{Z}^d$ by the rules (i)-(v) below. In general, these rules do not determine $\lambda(\cdot)$ uniquely.

(i) $\lambda(0, x)$ is the location of some initial B-particle, say $\lambda(0, x) = z_0$;

(ii) for all times s there is a distinguished B-particle, $\widehat{\rho}(s)$ say, at $\lambda(s, x)$; at time 0 we designate any of the B-particles at z_0 as $\widehat{\rho}(0)$;

(iii) $s \mapsto \lambda(s, x)$ can jump only at times when $\widehat{\rho}(s-)$ jumps away from $\lambda(s, x)$, and $\lambda(\cdot, x)$ is constant between such jumps;

(iv) if $\widehat{\rho}(s-)$ jumps from $\lambda(s-, x) = w$ to w' at some time s, and if this was the only particle at w at time $s-$, then $\lambda(\cdot, x)$ also jumps to w' at time s (so that $\lambda(s, x) = w'$) and $\widehat{\rho}(s) = \widehat{\rho}(s-)$, the particle which jumped at time s;

(v) if $\widehat{\rho}(s-)$ jumps from $\lambda(s-, x) = w$ to w' at some time s such that there is at least one other particle ρ' at w at time $s-$, then $\lambda(\cdot, x)$ jumps to w' at time s if and only if $\|w' - x\|_2 < \|w - x\|_2$, and in this case again $\widehat{\rho}(s) = \widehat{\rho}(s-)$; if, however, $\|w' - x\|_2 \geq \|w - x\|_2$, then $\lambda(\cdot, x)$ does not jump at time s and we take $\widehat{\rho}(s) = \rho'$.

We start with the distinguished particle being of type B, and by rules (iii)-(v), right after each jump of the distinguished particle, there still is a distinguished B-particle at the location of λ. From this it is easy to check recursively, from one jump of the distinguished particle to the next, that $\lambda(\cdot, x)$ automatically satisfies (ii). We merely have to note that if the distinguished particle has type B just before a jump, then all particles which coincide with the distinguished particle at that time also have type B.

Note that $\lambda(\cdot, x)$ can change only at the time at which the distinguished particle $\widehat{\rho}$ jumps (by rule (iii)). The next lemma shows that the jumps which occur at a time t when there are more than one particle at $\lambda(t)$ cause a certain drift of $\lambda(\cdot)$ towards x. We remind the reader that $\mathcal{F}_t^0 = \sigma$-field generated by $\{Y_s : s \leq t\}$ and that

$$\mathcal{F}_t = \cap_{h>0} \mathcal{F}_{t+h}^0.$$

We shall also use the following notation: P^σ denotes the conditional distribution of the process $\{Y_t\}_{t \geq 0}$ given that we start with $Y_0 = \sigma$; P without superscript is discussed just before Proposition 5.

$$I_1(u) = I[N_B\big(\lambda(u, x), u\big) = 1] = I[\widehat{\rho}(u) \text{ is the only particle present at } (\lambda(u, x), u)];$$

$$I_{\geq 2}(u) = I[N_B\big(\lambda(u, x), u\big) \geq 2]; \tag{4.51}$$

and with $e_{d+i} = -e_i$ for $1 \le i \le d$,

$$\Gamma_1(u) = \frac{1}{2d} \sum_{i=1}^{2d} \big[\|\lambda(u,x) + e_i - x\|_2 - \|\lambda(u,x) - x\|_2\big];$$
$$\Gamma_{\ge 2}(u) = \frac{1}{2d} \sum{}^* \big[\|\lambda(u,x) + e_i - x\|_2 - \|\lambda(u,x) - x\|_2\big], \tag{4.52}$$

where $\sum^*$ is the sum over those $i \in \{1, \dots, 2d\}$ for which

$$\|\lambda(u,x) + e_i - x\|_2 - \|\lambda(u,x) - x\|_2 < 0.$$

Lemma 10. *For any $\sigma \in \Sigma_0$*

$$M(t) = M(t,x) := \|\lambda(t,x) - x\|_2 - D_B \int_0^t \big[I_1(u)\Gamma_1(u) + I_{\ge 2}(u)\Gamma_{\ge 2}(u)\big] du \tag{4.53}$$

is a right continuous $\{\mathcal{F}_t\}$-martingale under the measure P^σ. $\{M(t)\}$ is also a right continuous $\{\mathcal{F}_t\}$-martingale under the measure $P = \int P\{Y_0 \in d\sigma\} P^\sigma$.

Proof. The last statement, about $\{M(t)\}$ under the measure P, is a consequence of the result under the measure P^σ. We therefore work only with the measure P^σ in this proof.

Right continuity of $\lambda(\cdot)$, and hence of $M(\cdot)$, is obvious from the right continuity of all particle paths. To check the martingale property it suffices to prove for all $A \in \mathcal{F}_t$ and $s > 0$ that

$$\int_A [M(t+s) - M(t)] dP^\sigma = 0. \tag{4.54}$$

To prove this we rewrite the left hand side as

$$\int_A dP^\sigma \sum_{k=1}^n \Big\{ \|\lambda(t + ks/n) - x\|_2 - \|\lambda(t + (k-1)s/n) - x\|_2$$
$$- D_B \int_A dP^\sigma \sum_{k=1}^n \int_{(k-1)s/n}^{ks/n} \Big[I_1(t+u-)\Gamma_1(t+u-) + I_{\ge 2}(t+u-)\Gamma_{\ge 2}(t+u-)\Big] du. \tag{4.55}$$

Note that the change from u to $u-$ in the argument of I_i or Γ_i has no influence on the integrals, because I_1, $I_{\ge 2}$, Γ_1 and $\Gamma_{\ge 2}$ can have a discontinuity only at one of the at most countably many times at which the distinguished particle jumps or at which a particle jumps to the position λ of the distinguished particle. Almost surely $[P^\sigma]$ there are only countably many jumps of the latter kind by virtue of (2.26)-(2.28) and the fact that at any time there are only finitely many B-particles (see the beginning of the proof of Lemma 2).

We now decompose the k-th summand in the first integral according to the number of jumps of the distinguished particle during $\big(t+(k-1)s/n, t+ks/n\big]$. Since $\lambda(t+ks/n)-\lambda(t+(k-1)s/n)=0$ if $\widehat\rho$ does not have a jump during $\big(t+(k-1)s/n, t+ks/n\big]$, it follows that the contribution of the k-the summand in the first integral is

$$\begin{aligned}\int_A dP^\sigma &\big[\|\lambda(t+ks/n)-x\|_2 - \|\lambda\big(t+(k-1)s/n\big)-x\|_2\big]\\ &\quad\times I[\widehat\rho \text{ has exactly one jump in } \big(t+(k-1)s/n, t+ks/n\big]\\ +\int_A dP^\sigma &\sum_{r\ge 2}\big[\|\lambda(t+ks/n)-x\|_2 - \|\lambda\big(t+(k-1)s/n\big)-x\|_2\big]\\ &\quad\times I[\widehat\rho \text{ has } r \text{ jumps in } \big(t+(k-1)s/n, t+ks/n\big].\end{aligned} \tag{4.56}$$

Since $\|\lambda(t+ks/n)-\lambda\big(t+(k-1)s/n\big)\|_2\le r$ if $\widehat\rho$ has r jumps during $\big(t+(k-1)s/n, t+ks/n\big]$, and since the proability of r such jumps is

$$e^{-D_B/n}\frac{1}{r!}\Big(\frac{D_B}{n}\Big)^r,$$

we see that the last integral in (4.56) is at most K_1n^{-2}, for some constant $K_1 = K_1(D_B)$.

Next we observe that the first integral in (4.56) can be expressed as an integral with respect to the time of the first jump of $\widehat\rho(t+(k-1)s/n)$ after $t+(k-1)s/n$. For the remainder of this proof denote the time and value of this jump by $\widehat\xi$ and $\widehat\upsilon$, respectively. Observe that $\widehat\rho(t+(k-1)s/n)$ is already of type B at time $t+(k-1)s/n$ (by rule (ii)), so that the increments of its path after time $t+(k-1)s/n$ are independent of $\mathcal{F}_{t+(k-1)s/n}$, as well as of the movement of all other particles after time $t+(k-1)s/n$. It follows that, conditionally on $\mathcal{F}_{t+(k-1)s/n}$, $\widehat\upsilon$ is independent of $\widehat\xi$ and $N_B\big(\lambda(\widehat\xi-),\widehat\xi-\big)$. If $N_B\big(\lambda(\widehat\xi-),\widehat\xi-\big)=1$, then the jump in λ at time $\widehat\xi$ equals $\widehat\upsilon$ by rule (iv), and the conditional expectation of $\|\lambda(\widehat\xi)-x\|_2-\|\lambda(\widehat\xi-)-x\|_2=\Gamma_1(\widehat\xi-)$ in this case. If $N_B\big(\lambda(\widehat\xi-),\widehat\xi-\big)\ge 2$, then by rule (v) $\lambda(\widehat\xi)-\lambda(\widehat\xi-)=\widehat\upsilon$ only if

$$\begin{aligned}\|\lambda(t+(k-1)s/n)+\widehat\upsilon-x\|_2 &= \|\lambda(\widehat\xi-)+\widehat\upsilon-x\|_2\\ &< \|\lambda(\widehat\xi-)-x\|_2 = \|\lambda(t+(k-1)s/n)-x\|_2.\end{aligned}$$

Otherwise λ does not change at $\widehat\xi$. It follows that given $\mathcal{F}_{t+(k-1)s/n}$ and $N_B\big(\lambda(\widehat\xi-),\widehat\xi-\big)\ge 2$, the conditional expectation of $\|\lambda(\widehat\xi)-x\|_2-\|\lambda(\widehat\xi-)-x\|_2$ equals $\Gamma_{\ge 2}(\widehat\xi-)$.

Therefore,

$$\begin{aligned}
&E^\sigma\big\{[\|\lambda(t+ks/n)-x\|_2-\|\lambda(t+(k-1)s/n)-x\|_2]\\
&\quad\times I[\widehat{\rho}\text{ has exactly one jump in }\big(t+(k-1)s/n,t+ks/n\big]\big|\mathcal{F}_{t+(k-1)s/n}\big\}\\
&=E^\sigma\Big\{\int_0^{s/n}\big[I_1(t+(k-1)s/n+u-)\Gamma_1(t+(k-1)s/n+u-)\\
&\qquad\qquad+I_{\geq 2}(t+(k-1)s/n+u-)\Gamma_{\geq 2}(t+(k-1)s/n+u-)\big]\\
&\quad\times P^\sigma\{\widehat{\xi}\in du,\widehat{\rho}\text{ has no jump in }(t+(k-1)s/n+u,t+ks/n]\}\big|\mathcal{F}_{t+(k-1)s/n}\Big\}\\
&=E^\sigma\Big\{\int_0^{s/n}e^{-D_Bu}\big[I_1(t+(k-1)s/n+u-)\Gamma_1(t+(k-1)s/n+u-)\\
&\qquad\qquad+I_{\geq 2}(t+(k-1)s/n+u-)\Gamma_{\geq 2}(t+(k-1)s/n+u-)\big]\\
&\quad\times e^{-D_B(s/n-u)}D_Bdu\big|\mathcal{F}_{t+(k-1)s/n}\Big\}.
\end{aligned}$$

Clearly this differs from

$$E^\sigma\Big\{\int_{(k-1)s/n}^{ks/n}\big[I_1(t+u-)\Gamma_1(t+u-)+I_{\geq 2}(t+u-)\Gamma_{\geq 2}(t+u-)\big]D_Bdu\big|\mathcal{F}_{t+(k-1)s/n}\Big\}.$$

by no more than K_2n^{-2}. The integral of this last expectation over A with respect to P^σ equals the k-th summand in the second integral in (4.55). It follows that the contributions of the k-th summands in the two integrals in (4.55) together is at most $[K_1+K_2]n^{-2}$. (4.54) follows if one sums over k and lets $n\to\infty$. ■

We now want to use known exponential bounds for large deviations of martingales with suitable bounds on their increments. The following lemma is a special case of estimates for discrete time (super) martingales with bounded jumps, such as can be found in [N], pp. 154-155 (see also the estimation of λ on p. 334 in [K]).

Lemma 11. *Assume that $\{\mathcal{G}_n\}_{n\geq 0}$ is an increasing sequence of σ-fields and that $D_n, n\geq 1$, are random variables which satisfy for all $n\geq 1$*

$$D_n \text{ is } \mathcal{G}_n\text{-measurable}, \tag{4.57}$$

$$|D_n|\leq c, \tag{4.58}$$

for some constant $0\leq c<\infty$, and

$$E\{D_n|\mathcal{G}_{n-1}\}=0. \tag{4.59}$$

Define $V_0=0$ and

$$V_n=\sum_{i=1}^n D_i,\quad A_n=\sum_{i=1}^n E\{D_n^2|\mathcal{G}_{n-1}\}$$

for $n \geq 1$. Then $\{V_n\}_{n\geq 0}$ is a $\{\mathcal{G}_n\}$-martingale and there exists a constant K_3, depending on c only, such that

$$P\{|V_n| \geq a + bA_n \text{ for some } n \geq 0\} \leq 2\exp[-K_3 ab], \quad a \geq 0,\, 0 \leq b \leq 1. \tag{4.60}$$

To deduce estimates for $M(t)$ from this lemma, we define $\sigma_0 = 0$ and for $k \geq 0$

$$\sigma_{k+1} = \min\big[\sigma_k + 1, \inf\{t > \sigma_k : t \text{ is a jump time of the distinguished particle } \widehat{\rho}\}\big].$$

We further take $D_n = M(\sigma_n) - M(\sigma_{n-1}), n \geq 1$, and $\mathcal{G}_n = \mathcal{F}_{\sigma_n}$. We then have $V_0 = 0$ and

$$V_n = M(\sigma_n, x) - M(0, x) = M(\sigma_n, x) - \|\lambda(0, x) - x\|_2 = M(\sigma_n, x) - \|z_0 - x\|_2,$$

with M given by (4.53). It is immediate from the definitions that

$$\sup_{\sigma_n \leq s \leq \sigma_{n+1}} |M(s) - M(\sigma_n)| \leq 1 + D_B. \tag{4.61}$$

Thus, (4.57)-(4.59) are satisfied with $c = 1 + D_B$. Moreover, $A_n \leq c^2 n$. Consequently,

$$\begin{aligned}
&P\{\big|M(\sigma_n) - M(0)\big| \geq a + bn \text{ for some } n \geq 0\} \\
&\leq P\{M(\sigma_n) - \sigma_0 \geq a + \frac{b}{c^2} A_n \text{ for some } n \geq 0\} \\
&\leq 2\exp[-K_3 abc^{-2}] \text{ if } b \leq c^2.
\end{aligned} \tag{4.62}$$

The jumptimes of $\widehat{\rho}$ are distributed like the jumptimes of a rate D_B Poisson process, so that

$$P\{\sigma_{\lfloor 2D_B t\rfloor} \leq t\} \leq \sum_{k \geq \lfloor 2D_B t\rfloor} e^{-D_B t}\frac{[D_B t]^k}{k!} \leq K_4 \exp[-K_5 t]. \tag{4.63}$$

Combined with (4.61) and (4.62) this shows that for $a \geq 2 + 2D_B$, $0 \leq b \leq 1$ and some constant $K_6 = K_6(D_B) > 0$

$$\begin{aligned}
P\{\sup_{s \leq t} |M(s) - M(0)| \geq a + bt\} &\leq P\{\sigma_{\lfloor 2D_B t\rfloor} \leq t\} \\
&+ P\{|M(\sigma_n) - M(0)| \geq a - 1 - D_B + \frac{b}{(1 + 2D_B)} n \text{ for some } n \leq 2D_B t\} \\
&\leq K_4 \exp[-K_5 t] + 2\exp[-K_6 ab].
\end{aligned} \tag{4.64}$$

In particular, if we take $K_6 \leq K_5$ (as we may), then we obtain for $a = bt, 0 \leq b \leq 1$ and $t \geq t_2 := 2(1 + D_B)/b$, that

$$P\{\sup_{s \leq t} |M(s) - M(0)| \geq 2bt\} \leq (2 + K_4)\exp[-K_6 b^2 t]. \tag{4.65}$$

Next we must find a lower bound for $\int_0^t [I_1(u)\Gamma_1(u) + I_{\geq 2}(u)\Gamma_{\geq 2}(u)]du$. Before we can do this we need a preparatory lemma. For $L \geq 2$ we define

$$\gamma(L,d) = \begin{cases} 1 & \text{if } d = 1 \\ [\log L]^{-1} & \text{if } d = 2 \\ L^{2-d} & \text{if } d \geq 3, \end{cases}$$

$$\mathcal{E}_n = \{\text{there is some particle } \rho' \neq \widehat{\rho}\big(3L^2(n-1)\big) \text{ in} \\ \lambda\big(3L^2(n-1), x\big) + [-L, L]^d \text{ at time } 3L^2(n-1)\}, \tag{4.66}$$

and

$$J_n = I\big[\widehat{\rho}(u) \text{ coincides with another particle} \\ \text{at some time } u \in (3L^2(n-1), L^2(3n-1)]\big].$$

Lemma 12. *There exists a constant $K_7 < \infty$, depending on d only, such that for all $4 \leq L^2 \leq t/4$*

$$E\{J_n | \mathcal{F}_{3L^2(n-1)}\} \geq K_7 \gamma^2(L,d) \text{ on the event } \mathcal{E}_n. \tag{4.67}$$

Proof. Fix an integer n and for brevity write m for $3L^2(n-1)$. Recall that the position of $\widehat{\rho}(m)$ is $\lambda(m)$. Suppose now that

$$\mathcal{E}_n \cap \{\rho' \text{ is already of type } B \text{ at time } m\} \tag{4.68}$$

occurs. Then ρ' and $\widehat{\rho}(m)$ continue after time m to perform random walks $\{S'\}$ and $\{S''\}$ which are copies of S^B, and which are independent of each other and all other particles. Since, on the event $\mathcal{E}_n$, these two particles have a distance at most $L\sqrt{d}$ from each other at time m, we can use standard random walk estimates to find a lower bound for

$$P\{\widehat{\rho}(m) \text{ and } \rho' \text{ coincide at some time } u \in (m, m+L^2] | \mathcal{F}_m\} \\ = P\{S'_u - S''_u = z \text{ for some } u \leq L^2\},$$

where $z = -\pi(m, \rho') + \pi\big(m, \widehat{\rho}(m)\big)$. Indeed,

$$\int_{u \leq L^2} P\{S'_u - S''_u = z\} du \\ = E\{\text{amount of time during } [0, L^2] \text{ with } S'_u - S''_u = z\} \\ = \int_{s \leq L^2} P\{\text{smallest } u \text{ with } S'_u - S''_u = z \text{ lies in } ds\} \\ \times E\{\text{amount of time during } [0, L^2 - s] \text{ with } S'_u - S''_u = \mathbf{0}\} \\ \leq P\{S'_u - S''_u = z \text{ for some } u \leq L^2\} \int_{u \leq L^2} P\{S'_u - S''_u = \mathbf{0}\} du.$$

The integrals in the extreme left and right hand sides here can be estimated by means of the local central limit theorem to obtain that

$$P\{\widehat{\rho}(m) \text{ and } \rho' \text{ coincide at some time } u \in (m, m+L^2] | \mathcal{F}_m\} \geq K_8 \gamma(L, d) \quad (4.69)$$

(see Theorem 2.2 in [AMP] or Lemmas 5.1 and 5.2 in [CD] for similar arguments). Note, however, that the particular distinguished particle $\widehat{\rho}(m)$ may be different from the distinguished particle $\widehat{\rho}(u)$ at a time $u > m$. Thus the fact that $\widehat{\rho}(m)$ and ρ' coincide at time u does not imply that ρ' is at $\lambda(u)$ at time u (or, equivalently, $I_{\geq 2}(u) = 1$). Nevertheless, this can fail only if rule (v) was invoked at some time during $(m, u]$, which means that there must have been some time $u' \in (m, u]$ at which $\widehat{\rho}(u')$ coincided with another particle. It follows that $J_n = 1$ on the event (4.68) intersected with the event in (4.69). Thus, (4.69) shows that $P\{J_n = 1 | \mathcal{F}_m\} \geq K_8 \gamma(L, d)$ on the event (4.68).

To deal with the event

$$\mathcal{E}_n \cap \{\rho' \text{ is of type } A \text{ at time } m\}, \quad (4.70)$$

we set $\pi^*(s) = \pi(0, \rho') + \pi_A(s, \rho')$ for $s \geq m$. This is the path which ρ' would follow in the system $\mathcal{P}^*$ introduced just after (2.16), i.e., the system without B-particles. Now we first use that (4.69) can be strengthened to

$$\begin{aligned} P\big\{&\text{there is some time } u \in (m, m+L^2] \text{ such that } \widehat{\rho}(m) \text{ and} \\ &\pi^*(u) \text{ coincide at } u \text{ and such that } \widehat{\rho}(m) \in \lambda(m) + [-K_9 L, K_9 L]^d \\ &\text{as well as } \rho' \in \lambda(m) + [-K_9 L, K_9 L]^d \text{ for } m \leq s \leq u] \big| \mathcal{F}_m \big\} \\ &\geq K_{10} \gamma(L, d) \text{ on } \mathcal{E}_n, \end{aligned} \quad (4.71)$$

still by standard random walk estimates (which we omit). If $\widehat{\rho}(m)$ and π^* coincide at time u, and

$$\rho' \text{ has not met any } B\text{-particle during } (m, u), \quad (4.72)$$

then $\widehat{\rho}(m)$ and ρ' actually coincide at time u, and by the argument following (4.69) we see that then $I_{\geq 2}(u') = 1$ at some time $u' \in (m, m+L^2]$. On the other hand, if the event in (4.71) occurs, but not the event in (4.72), then the first time at which ρ' turns into a B-particle, $\theta(\rho')$, equals $u \in [m, m+L^2]$, and at that time both $\widehat{\rho}(m)$ and ρ' lie in $\lambda(m) + [-K_9 L, K_9 L]^d$. Conditionally on $\mathcal{F}_{\theta(\rho')}$ we then have a probability of at least $K_{11} \gamma(L, d) > 0$ that $\widehat{\rho}(m)$ and ρ' coincide during $(\theta(\rho'), \theta(\rho') + L^2] \subset (m, m+2L^2]$, for the same reasons as for (4.69). As in the lines following (4.69) this then implies that $\widehat{\rho}(u')$ coincides with ρ' at some time

$u' \in (m, m+2L^2]$. It follows from these estimates that on the event (4.70) it holds

$$\begin{aligned}
&P\{J_n = 1|\mathcal{F}_m\}\\
&\qquad \geq P\{\text{the events in (4.71) and (4.72) occur}|\mathcal{F}_m\}\\
&\qquad\quad + E\Big\{I[\text{the event in (4.71) occurs, but the event in (4.72) does not}]\\
&\qquad\qquad\quad \times P\{\widehat{\rho}(m) \text{ and } \rho' \text{ coincide during } [\theta(\rho'), \theta(\rho') + L^2]|\mathcal{F}_{\theta(\rho')}\}\Big|\mathcal{F}_m\Big\}\\
&\qquad \geq P\{\text{the events in (4.71) and (4.72) occur}|\mathcal{F}_m\}\\
&\qquad\qquad\quad + P\{\text{the event in (4.71) occurs, but the event}\\
&\qquad\qquad\qquad \text{in (4.72) does not}|\mathcal{F}_m\}K_{11}\gamma(L,d)\\
&\qquad \geq K_{11}\gamma(L,d)P\{\text{the event in (4.71) occurs}|\mathcal{F}_m\}\\
&\qquad \geq K_{11}K_{10}\gamma^2(L,d).
\end{aligned}$$

Thus (4.67) holds, both on (4.68) and on (4.70). ■

We next derive a lower bound for

$$Z(t) = Z(t,x) := \int_0^t I_{\geq 2}(u)du.$$

in terms of

$$\begin{aligned}
V(t,L) = V(t,L,x) := \sum_{1 \leq n \leq L^{-2}t/3} I\Big[&\text{there is some particle other than}\\
&\widehat{\rho}\big(3L^2(n-1)\big) \text{ inside } \lambda(3L^2(n-1),x) + [-L,L]^d \text{ at time } 3L^2(n-1)\Big].
\end{aligned} \tag{4.73}$$

Lemma 13. *For* $0 < \varepsilon \leq 1$ *and* $L \geq 1$

$$\begin{aligned}
&P\{Z(t) \leq \varepsilon\gamma^2(L,d)L^{-2}t\}\\
&\qquad \leq P\Big\{V(t,L) \leq \frac{2\varepsilon}{K_7}e^{2D_B}L^{-2}t\Big\} + 2\exp\Big[-\frac{K_3}{3}\varepsilon^2\gamma^4(L,d)L^{-2}t\Big].
\end{aligned} \tag{4.74}$$

Proof. We define

$$\mathcal{G}_n = \mathcal{F}_{3L^2n},$$

$$\widetilde{J}_n = \min\Big\{1, \int_{3L^2(n-1)}^{3L^2n} I_{\geq 2}(u)du\Big\}$$

and

$$D_n = \widetilde{J}_n - E\{\widetilde{J}_n|\mathcal{G}_{n-1}\}.$$

Note that $0 \le \widetilde{J}_n \le 1$, so that $|D_n| \le 1$, $E\{D_n^2|\mathcal{G}_{n-1}\} \le 1$ and

$$A_{\lfloor L^{-2}t/3\rfloor} := \sum_{1\le n\le L^{-2}t/3} E\{D_n^2|\mathcal{G}_{n-1}\} \le L^{-2}t/3.$$

Therefore, (4.60) with $c = 1$,

$$a = \varepsilon\gamma^2(L,d)L^{-2}t/3 \text{ and } b = \varepsilon\gamma^2(L,d)$$

yields

$$P\Big\{\Big|\sum_{1\le n\le L^{-2}t/3}\big[\widetilde{J}_n - E\{\widetilde{J}_n|\mathcal{G}_{n-1}\}\big]\Big| \ge \varepsilon\gamma^2(L,d)L^{-2}t\Big\}$$
$$\le 2\exp\big[-\frac{K_3}{3}\varepsilon^2\gamma^4(L,d)L^{-2}t\big].$$

In particular,

$$\begin{aligned}
&P\{Z(t) \le \varepsilon\gamma^2(L,d)L^{-2}t\}\\
&\le P\Big\{\sum_{1\le n\le L^{-2}t/3}\widetilde{J}_n \le \varepsilon\gamma^2(L,d)L^{-2}t\Big\}\\
&\le P\Big\{\sum_{1\le n\le L^{-2}t/3}E\{\widetilde{J}_n|\mathcal{G}_{n-1}\} \le 2\varepsilon\gamma^2(L,d)L^{-2}t\Big\}\\
&\qquad + P\Big\{\Big|\sum_{1\le n\le L^{-2}t/3}\big[\widetilde{J}_n - E\{\widetilde{J}_n|\mathcal{G}_{n-1}\}\big]\Big| \ge \varepsilon\gamma^2(L,d)L^{-2}t\Big\}\\
&\le P\Big\{\sum_{1\le n\le L^{-2}t/3}E\{\widetilde{J}_n|\mathcal{G}_{n-1}\} \le 2\varepsilon\gamma^2(L,d)L^{-2}t\Big\} + 2\exp\big[-\frac{K_3}{3}\varepsilon^2\gamma^4(L,d)L^{-2}t\big].
\end{aligned} \tag{4.75}$$

Finally, we observe that on the event $\mathcal{E}_n$ (see (4.66))

$$\begin{aligned}
E\{\widetilde{J}_n|\mathcal{G}_{n-1}\} \ge P\big\{&\widehat{\rho}(u) \text{ coincides with another particle } \rho' \text{ at some time}\\
&u \in (3L^2(n-1), L^2(3n-1)] \text{ and the positions of } \widehat{\rho}(u') \text{ and}\\
&\rho' \text{ and } \lambda(u',x) \text{ all stay together for } u' \in [u,u+1]|\mathcal{G}_{n-1}\}\\
\ge{}& \exp[-2D_B]P\{J_n = 1\big|\mathcal{G}_{n-1}\}\\
\ge{}& \exp[-2D_B]K_7\gamma^2(L,d) \text{ (by (4.67))}.
\end{aligned}$$

The lemma now follows from

$$\sum_{1\le n\le L^{-2}t/3}E\{\widetilde{J}_n|\mathcal{G}_{n-1}\} \ge \exp[-2D_B]K_7\gamma^2(L,d)\sum_{1\le n\le L^{-2}t/3}I[\mathcal{E}_n]$$
$$= \exp[-2D_B]K_7\gamma^2(L,d)V(t,L).$$

■

The next lemma gives an upper bound for $\int_0^t [I_1(u)\Gamma_1(u) + I_{\ge 2}(u)\Gamma_{\ge 2}(u)]du$ in terms of $Z(t)$.

Lemma 14. *There exist constants* $0 < K_{12}, K_{13}, K_{14} < \infty$, *which depend on* d *and* D_B *only, such that for all* $z > 0$

$$\int_0^t [I_1(u)\Gamma_1(u) + I_{\geq 2}(u)\Gamma_{\geq 2}(u)]du \\ \leq \frac{K_{12}\gamma^2(L,d)}{zL^2}t + K_{12}\int_0^t I\Big[\|\lambda(u) - x\|_2 \leq \frac{zL^2}{\gamma^2(L,d)}\Big]du - K_{13}Z(t). \tag{4.76}$$

Consequently, for

$$0 < \varepsilon \leq 1, \quad \|x - z_0\|_2 \leq \frac{K_{13}\varepsilon D_B \gamma^2(L,d)}{4L^2}t, \quad z = \frac{4K_{12}}{K_{13}\varepsilon}, \tag{4.77}$$

$$L \geq L_0 := \Big[\frac{\varepsilon D_B K_{13}}{8}\Big]^{1/2} \vee 3, \tag{4.78}$$

and for $t \geq L^2[\varepsilon\gamma^2(L,d)]^{-1}t_3$ *for some* $t_3 = t_3(D_B)$ *it holds*

$$P\Big\{\int_0^t I\Big[\|\lambda(u) - x\|_2 \leq \frac{zL^2}{\gamma^2(L,d)}\Big]du \leq \frac{K_{13}\varepsilon\gamma^2(L,d)}{4K_{12}L^2}t\Big\} \\ \leq (2 + K_4)\exp[-K_{14}\varepsilon^2\gamma^4(L,d)L^{-4}t] + P\big\{V(t,L) \leq \frac{2\varepsilon}{K_7}e^{2D_B}L^{-2}t\big\} \\ + 2\exp\big[-\frac{K_3}{3}\varepsilon^2\gamma^4(L,d)L^{-2}t\big]. \tag{4.79}$$

Proof. We shall show by simple calculus that there exist some constants K_{12}, K_{13} which depend on d only, such that for $\lambda, x \in \mathbb{Z}^d$

$$\frac{1}{2d}\sum_{i=1}^{2d}\big[\|\lambda + e_i - x\|_2 - \|\lambda - x\|_2\big] \leq \frac{K_{12}}{\|\lambda - x\|_2 + 1}, \tag{4.80}$$

and, with $\sum^*$ as in (4.52),

$$\frac{1}{2d}\sum{}^*\big[\|\lambda + e_i - x\|_2 - \|\lambda - x\|_2\big] \leq -K_{13} + \frac{K_{12}}{\|\lambda - x\|_2 + 1}. \tag{4.81}$$

Moreover the left hand sides of (4.80) and (4.81) are at most 1 in absolute value.

Before we prove these inequalities we show that they imply the lemma. Indeed, it follows from (4.80),(4.81) and the definitions (4.52) that the left hand side of (4.76) is at most

$$\frac{K_{12}\gamma^2(L,d)}{zL^2}\int_0^t \big(I_1(u) + I_{\geq 2}(u)\big)I\Big[\|\lambda(u) - x\|_2 > \frac{zL^2}{\gamma^2(L,d)}\Big]du \\ + K_{12}\int_0^t \big(I_1(u) + I_{\geq 2}(u)\big)I\Big[\|\lambda(u) - x\|_2 \leq \frac{zL^2}{\gamma^2(L,d)}\Big]du - K_{13}\int_0^t I_{\geq 2}(u)du \\ \leq \frac{K_{12}\gamma^2(L,d)}{zL^2}t + K_{12}\int_0^t I\Big[\|\lambda(u) - x\|_2 \leq \frac{zL^2}{\gamma^2(L,d)}\Big]du - K_{13}Z(t).$$

This proves (4.76).

To prove (4.79) we take $b = \varepsilon D_B K_{13}\gamma^2(L,d)/(8L^2)$ in (4.65). For $L \ge L_0$ this b satisfies $b \le 1$. Then we obtain, by means of (4.76), that outside a set of probability at most $(2+K_4)\exp[-K_6 b^2 t]$ it holds

$$\begin{aligned} 0 \le &\|\lambda(t) - x\|_2 \le \|\lambda(0) - x\|_2 + 2bt + \frac{D_B K_{12}\gamma^2(L,d)}{zL^2} t \\ &+ D_B K_{12}\int_0^t I\Big[\|\lambda(u) - x\|_2 \le \frac{zL^2}{\gamma^2(L,d)}\Big] du - D_B K_{13} Z(t) \end{aligned}$$

for $t \ge t_2 = L^2[\varepsilon\gamma^2(L,d)]^{-1} t_3$ for some $t_3 = t_3(D_B)$. By substitution of the chosen values of x, b and z this yields

$$\int_0^t I\Big[\|\lambda(u) - x\|_2 \le \frac{zL^2}{\gamma^2(L,d)}\Big] du \ge \frac{K_{13}}{K_{12}} Z(t) - \frac{3K_{13}\varepsilon\gamma^2(L,d)}{4K_{12}L^2} t. \qquad (4.82\ )$$

If we exclude a further set of probability at most equal to the right hand side of (4.74), then the right hand side of (4.82) exceeds $K_{13}\varepsilon\gamma^2(L,d)[4K_{12}L^2]^{-1}t$. Thus (4.79) also follows from (4.80) and (4.81).

We turn to the proof of (4.80) and (4.81). The sentence following (4.81) is trivial. We can therefore adjust K_{12} so that (4.80) and (4.81) are valid on any given finite set of values for $\|\lambda - x\|_2$. In particular, we may restrict ourselves to proving (4.80), (4.81) for $\|\lambda - x\|_2 \ge 2$. Now the Taylor expansion

$$\|a+b\|_2 = \sqrt{\|a\|_2^2 + 2a\cdot b + \|b\|_2^2} = \|a\|_2 + \frac{2a\cdot b + \|b\|_2^2}{2\|a\|_2} + O\Big(\frac{\|a\|_2^2\|b\|_2^2 + \|b\|_2^4}{\|a\|_2^3}\Big)$$

shows that the left hand side of (4.80) equals

$$\frac{1}{2d}\sum_{i=1}^{2d} \frac{(\lambda - x)\cdot e_i}{\|\lambda - x\|_2} + \frac{H(\lambda - x)}{\|\lambda - x\|_2} = \frac{H(\lambda - x)}{\|\lambda - x\|_2} \qquad (4.83)$$

for some function H which is bounded on $\{\lambda \ne x\} = \{\|\lambda - x\|_2 \ge 1\}$ (recall that $\lambda, x \in \mathbb{Z}^d$). Thus (4.80) holds.

For (4.81) we write $\lambda - x = \sum_{i=1}^d n_i e_i$, with integer coefficients n_i (since $\lambda - x \in \mathbb{Z}^d$). Then for a given $i \in \{1,\dots,d\}$ there are three possibilities: $n_i > 0, n_i < 0, n_i = 0$. If $n_i > 0$, and hence $n_i \ge 1$, then $d+i$ is contained in $\sum^*$, but not i. Thus in this case, $(2d)$ times the contribution of the term with $d+i$ to the left hand side of (4.81) is

$$\begin{aligned} &\Big[\sum_{k\ne i} n_k^2 + (n_i - 1)^2\Big]^{1/2} - \Big[\sum_{k\ne i} n_k^2 + n_i^2\Big]^{1/2} \\ &= \Big\{\Big[\sum_{k\ne i} n_k^2 + (n_i - 1)^2\Big]^{1/2} + \Big[\sum_{k\ne i} n_k^2 + n_i^2\Big]^{1/2}\Big\}^{-1}(-2n_i + 1) \\ &\le -\frac{1}{2}\Big[\sum_{k\ne i} n_k^2 + n_i^2\Big]^{-1/2} n_i. \end{aligned}$$

(In the last inequality we used that $2n_i - 1 \geq n_i$ and that the term in the denominator with n_i^2 exceeds that with $(n_i-1)^2$.) If $n_i < 0$, then only i is contained in $\sum^*$, but not $d+i$, and the preceding estimates hold with n_i replaced by $-n_i$. Finally, if $n_i = 0$, then neither i nor $d+i$ are contained in $\sum^*$. Thus the left hand side of (4.81) is at most

$$-\frac{1}{4d}\Big[\sum_{k=1}^{d} n_k^2\Big]^{-1/2}\sum_{i=1}^{d}|n_i| \leq -\frac{1}{4d}I[\lambda - x \neq 0]. \qquad \blacksquare$$

Proof of Theorem 2. We now have everything in place to prove Theorem 2. Fix $K > 0$ and a large t. We first use that the distinguished particle $\widehat{\rho}$ jumps with the constant rate D_B. Therefore, it holds for any x

$$\begin{aligned}&P\{\lambda(\cdot,x) \text{ has more than } 2D_Bt \text{ jumps during } [0,t]\}\\ &\leq P\{\widehat{\rho}(\cdot) \text{ has more than } 2D_Bt \text{ jumps during } [0,t]\}\\ &\leq K_4\exp[-K_5t] \text{ (see (4.63))}.\end{aligned}$$

Note that if $\lambda(\cdot,x)$ has no more than $2D_Bt$ jumps during $[0,t]$, then also $\|\lambda(s,x) - \lambda(0,x)\| = \|\lambda(s,x) - z_0\|_\infty \leq 2D_Bt$ for $s \leq t$. Thus, for sufficiently large t (see (4.3) for Ξ)

$$P\{\{\lambda(s,x)\}_{s\leq t} \notin \bigcup_{\ell \leq 2D_Bt}\Xi(\ell,t) \text{ for any } x \text{ with } \|x\| \leq t\} \leq K_{15}t^d\exp[-K_5t]. \tag{4.84}$$

Proposition 9 (with K replaced by $K+d$) now tells us, that outside a further set of probability at most $2/t^{K+d}$, we have for $r \geq r_0, \ell \leq 2D_Bt$ and $t \geq t_1$ that $\Phi_r(\ell) \leq \varepsilon_0 C_0^{-6r}(1+2D_B)t$. Therefore, if we write $\widetilde{\lambda}(s,x)$ for the space-time point $\lambda(s,x) \times \{s\}$, then

$$\begin{aligned}&\text{each path } \{\widetilde{\lambda}(s,x)\}_{s\leq t} \text{ with } \|x\| \leq t \text{ intersects at most}\\ &\qquad\qquad C_0^{-6r}\varepsilon_0(1+2D_B)t \text{ bad } r\text{-blocks}\end{aligned} \tag{4.85}$$

(see (4.9) for the definition of a bad block). We choose $\varepsilon_0 > 0$ such that

$$\varepsilon_0(1+2D_B) < \frac{1}{7}, \tag{4.86}$$

and then we fix r at some value $r_1 \geq r_0$ such that

$$\gamma_{r_1}\mu_A C_0^{dr_1} \geq 2,\ C_0^{6r_1} \geq \frac{3D_BK_{13}}{8}, \tag{4.87}$$

and

$$\frac{2\varepsilon_0}{K_7}e^{2D_B} < \frac{1}{6} \tag{4.88}$$

((4.87) is possible because $C_0 \geq 2$ and $\gamma_{r_1} \geq \gamma_0 > 0$). We claim that with this choice, (4.85) implies that

$$\begin{aligned}&\text{for each path } \{\lambda(s,x)\}_{s\leq t} \text{ with } \|x\| \leq t \text{ there are at least}\\ &(1/6)C_0^{-6r_1}t \text{ integers } n \text{ for which there exists a particle}\\ &\rho' \neq \widehat{\rho} \text{ inside } \lambda(3C_0^{6r}n,x) + [-C_0^{3r_1}, C_0^{3r_1}] \text{ at time } 3C_0^{6r_1}n.\end{aligned} \tag{4.89}$$

To see this recall that $\Delta_r = C_0^{6r}$, and note that each point $\widetilde{\lambda}(kC_0^{6r_1},x)$ belongs to a unique r_1-block $\mathcal{B}_{r_1}(\mathbf{i},k)$, and for different k, these blocks are disjoint. Thus for each x, $\{\lambda(s,x)\}_{s\leq t}$ intersects at least $\lfloor C_0^{-6r_1}t\rfloor$ distinct r_1-blocks. If (4.85) holds, then, by (4.86), at most $(1/7)C_0^{-6r_1}t$ of these blocks are bad, so that for large t there are at least $(6/7)C_0^{-6r_1}t - 1$ values of $k \leq \lfloor C_0^{-6r_1}t\rfloor$ such that $\widetilde{\lambda}(kC_0^{6r_1},x)$ belongs to a good r_1-block. At least $(1/6)C_0^{-6r_1}t$ of these will have k divisible by 3, say $k = 3n$, with $n \leq \lfloor C_0^{-6r_1}t\rfloor/3$. If $\widetilde{\lambda}(3C_0^{6r_1}n,x)$ belongs to a good r_1-block, then by definition

$$U_{r_1}(\lambda(3C_0^{6r_1}n,x), 3C_0^{6r_1}n) = \sum_{y\in\mathcal{Q}_r(\lambda^*)} N^*(y, 3C_0^{6r_1}n) \geq \gamma_{r_1}\mu_A C_0^{dr_1} \geq 2$$

(see (4.87)), where we have temporarily written λ^* for $\lambda(3C_0^{6r_1}n,x)$. In particular, there have to be two particles in $\prod_{s=1}^d [\lambda^*(s), \lambda^*(s)+C_0^r)$ at time $3C_0^{6r_1}n$, and one of these must be different from the distinguished particle at $\lambda(3C_0^{6r_1}n)$. This justifies our claim (4.89).

In the notation of (4.73) the preceding paragraph shows that (4.85)-(4.86) imply that for all x with $\|x\| \leq t$ and

$$L = C_0^{3r_1},$$

it holds

$$V(t,L,x) \geq (1/6)C_0^{-6r_1}t > \frac{2\varepsilon_0}{K_7}e^{2D_B}C_0^{-6r_1}t = \frac{2\varepsilon_0}{K_7}e^{2D_B}L^{-2}t.$$

Thus the bound (4.84) and the lines following it prove (for large t)

$$P\{V(t,L,x) < \frac{2\varepsilon_0}{K_7}e^{2D_B}L^{-2}t\} \leq K_{15}t^d\exp[-K_5t] + 2t^{-K-d} \leq 3t^{-K-d}.$$

(4.79) with $\varepsilon = \varepsilon_0$ and z as in (4.77) then shows that for large t and for all x with

$$\|x - z_0\|_2 \leq \frac{K_{13}\varepsilon_0 D_B\gamma^2(L,d)}{4L^2}t \text{ and } \|x\| \leq t, \tag{4.90}$$

$$P\Big\{\int_0^t I\Big[\|\lambda(u) - x\|_2 \leq \frac{4K_{12}L^2}{K_{13}\varepsilon_0\gamma^2(L,d)}\Big]du \leq \frac{K_{13}\varepsilon_0\gamma^2(L,d)}{4K_{12}L^2}t\Big\} \leq 4t^{-K-d}. \tag{4.91}$$

Now fix an x and assume that

$$\int_0^t I\Big[\|\lambda(u) - x\|_2 \le \frac{4K_{12}L^2}{K_{13}\varepsilon_0\gamma^2(L,d)}\Big]du > \frac{K_{13}\varepsilon_0\gamma^2(L,d)}{4K_{12}L^2}t. \tag{4.92}$$

This implies trivially that $\|\lambda(u,x)-x\|_2 \le 4K_{12}L^2/\big(K_{13}\varepsilon_0\gamma^2(L,d)\big)$ for some $u \le t$. However, we want more. For Theorem 2 we want that for each $x \in \mathcal{C}(C_2 t)$ there exists a $u \le t$ at which x is visited by a B-particle. To show that such a u exists with high probability, we define the stopping times

$$\begin{aligned} u_0 = 0, u_{i+1} &= u_{i+1}(x) \\ &= \inf\Big\{u \ge u_i + L^4\gamma^{-4}(L,d) : \|\lambda(u,x) - x\|_2 \le \frac{4K_{12}L^2}{K_{13}\varepsilon_0\gamma^2(L,d)}\Big\}. \end{aligned}$$

(4.92) implies that $u_i + L^4\gamma^{-4}(L,d) \le t$ for at least

$$\chi := \Big\lceil \frac{K_{13}\varepsilon_0\gamma^6(L,d)}{4K_{12}L^6}t\Big\rceil - 1$$

values of $i \ge 1$ with $u_i \le t$. By definition $u_{i+1} - u_i \ge L^4\gamma^{-4}(L,d)$ and by the right continuity of $\lambda(\cdot)$

$$\|\lambda(u_i,x) - x\|_2 \le \frac{4K_{12}L^2}{K_{13}\varepsilon_0\gamma^2(L,d)} =: K_{16}\frac{L^2}{\gamma^2(L,d)}. \tag{4.93}$$

Define

$$\widetilde{L} = \frac{L^2}{\gamma^2(L,d)}.$$

Then the same random walk estimates as for (4.69) give that

$$\begin{aligned} P\big\{&\widehat{\rho}(u_i) \text{ visits } x \text{ at some time in } (u_i,u_{i+1}]|\mathcal{F}_{u_i}\big\} \\ &\ge K_{17}\gamma(\widetilde{L},d) \ge \begin{cases} K_{17} & \text{if } d=1 \\ K_{17}[4\log L]^{-1} & \text{if } d=2 \\ K_{17}L^{(d-1)(4-2d)} & \text{if } d \ge 3. \end{cases} \end{aligned} \tag{4.94}$$

Note that we are estimating here the probability that the distinguished particle of time u_i visits x at some time u, rather than that $\lambda(u,x) = x$. But conditionally on $\mathcal{F}_{u_i}$, the B-particle $\widehat{\rho}(u_i)$ performs a random walk which is a copy of S^B, and this fact is the basis for the estimate (4.94). Finally we apply Lemma 11 once more. We take $\mathcal{G}_n = \mathcal{F}_{u_n}$,

$$H_n := I[\widehat{\rho}(u_{n-1}) \text{ visits } x \text{ at some time in } (u_{n-1},u_n].$$

and

$$D_n = H_n - E\{H_n|\mathcal{G}_{n-1}\}.$$

Since H_n takes on only the values 0 or 1, it is easy to see from (4.94) that on the event (4.92)

$$\sum_{n=1}^{\chi} E\{H_n|\mathcal{G}_{n-1}\} \geq \chi K_{17}\gamma(\widetilde{L},d) \geq K_{18}\gamma(\widetilde{L},d)\frac{\gamma^6(L,d)}{L^6}t$$

and

$$A_\chi = \sum_{n=1}^{\chi} E\{D_n^2|\mathcal{G}_{n-1}\} \leq K_{19}\gamma(\widetilde{L},d)\frac{\gamma^6(L,d)}{L^6}t.$$

It is then easy to deduce from Lemma 11, with $a = (1/2)K_{18}\gamma(\widetilde{L},d)\frac{\gamma^6(L,d)}{L^6}t$, $b = K_{18}/(2K_{19}) \wedge 1$ and $c = 1$, that for x satisfying (4.90) and t large

$$\begin{aligned}
&P\{x \text{ is not visited by a } B\text{-particle by time } t\} \\
&\leq P\{(4.92) \text{ does not occur}\} + P\{(4.92) \text{ occurs, but } \sum_{i=1}^{\chi} H_n = 0\} \\
&\leq 4t^{-K-d} \\
&\quad + P\Big\{(4.92) \text{ occurs, but } \sum_{i=1}^{\chi}\big[H_n - E\{H_n|\mathcal{G}_{n-1}\}\big] \leq -K_{18}\gamma(\widetilde{L},d)\frac{\gamma^6(L,d)}{L^6}t\Big\} \\
&\leq 4t^{-K-d} + P\{\sum_{n=1}^{\chi}\big[H_n - E\{H_n|\mathcal{G}_{n-1}\big] \leq -a - bA_\chi\} \\
&\leq 4t^{-K-d} + 2\exp\big[-K_{20}\gamma(\widetilde{L},d)\frac{\gamma^6(L,d)}{L^6}t\big] \\
&\leq 5t^{-K-d}.
\end{aligned} \tag{4.95}$$

Theorem 2 with

$$C_2 = \frac{K_{13}\varepsilon_0 D_B \gamma^2(L,d)}{8L^2\sqrt{d}} \wedge 1$$

now follows by summing (4.95) over all x which satisfy (4.90). ■

5. Proof of Proposition 3.

A basic step for the proof is a monotonicity property which is proven via a coupling argument. We formulate it as a separate lemma.

Lemma 15. *Assume $D_A = D_B$ and let $\sigma^{(2)} \in \Sigma_0$. Assume further that $\sigma^{(1)}$ lies below $\sigma^{(2)}$ in the following sense:*

for any site $z \in \mathbb{Z}^d$, all particles present in $\sigma^{(1)}$ at z are also present in $\sigma^{(2)}$ at z, (5.1)

and

at any site z at which the particles in $\sigma^{(2)}$ have type A,
the particles also have type A in $\sigma^{(1)}$. (5.2)

Let $\pi_A(\cdot,\rho) = \pi_B(\cdot,\rho)$ be the random walk paths associated to the various particles and assume that the Markov processes $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ are constructed by means of the same set of paths $\pi_A(\cdot,\rho) = \pi_B(\cdot,\rho)$ and starting with state $\sigma^{(1)}$ and $\sigma^{(2)}$, respectively (as defined in Section 2). Then, almost surely, $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ satisfy (5.1) and (5.2) for all t with $\sigma^{(i)}$ replaced by $Y_t^{(i)}$, $i = 1,2$. In particular, $\sigma^{(1)} \in \Sigma_0$ and (2.20) holds almost surely for $\{Y_t^{(1)}\}$.

Proof. Couple the processes $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ as in the statement of the lemma. Specifically, first choose independent paths $s \mapsto \pi_A(s,\rho)$ for all particles ρ present in $\sigma^{(2)}$ and construct $\{Y_t^{(2)}\}$ with the help of these paths (as in (2.6) and (2.7), with $\pi_B(s,\rho) = \pi_A(s,\rho)$ for all s,ρ). We then assign to each particle ρ present in $\sigma^{(1)}$ the same path $s \mapsto \pi_A(s,\rho)$ as assigned to ρ in $\sigma^{(2)}$. By (5.1) this assigns a path to each particle present in $\sigma^{(1)}$. We then construct $\{Y_t^{(1)}\}$ on the basis of these paths. The position at time t of a particle ρ starting at z is then $z + \pi_A(t,\rho)$, in whichever of the systems the particle is present. It is immediate from this that (5.1) with $\sigma^{(i)}$ replaced by $Y_t^{(i)}$, $i = 1,2$, is valid.

To show (5.2) with $\sigma^{(i)}$ replaced by $Y_t^{(i)}$, $i = 1,2$, we first note that (2.20) a.s. holds for $\{Y_t^{(2)}\}$, because $\sigma^{(2)} \in \Sigma_0$. Then, by (5.1), a.s. (2.20) holds in both systems. Now let $\tau_0^{(i)} = 0$ and for $k \geq 1$, let $\tau_k^{(i)}$ be the k-th time at which a new particle changes from type A to type B in $\{Y_t^{(i)}\}$. More formally, as in (2.3), $\tau_{k+1}^{(i)} = \inf\{t > \tau_k^{(i)} :$ a B-particle coincides with an A particle at time t in $\{Y_t^{(i)}\}\}$. We shall show by induction on $k \geq 0$ that at each time $\tau_k^{(1)}$ the property (5.2) with $\sigma^{(i)}$ replaced by $Y_{\tau_k^{(1)}}^{(i)}$, still holds, and there are only finitely many B-particles in both systems at time $\tau_k^{(1)}$. We may restrict ourselves to sample points for which $\min(\widehat{\tau}, \tau_\infty) = \infty$ in the $\{Y_t^{(2)}\}$-system, because $\sigma^{(2)} \in \Sigma_0$. Assume then that at time $\tau_k^{(1)}$, (5.2) with $\sigma^{(i)}$ replaced by $Y_{\tau_k^{(1)}}^{(i)}$ still holds. Since the second system has only finitely many B-particles at time $\tau_k^{(1)}$, this, together with (5.2) with $\sigma^{(i)}$ replaced by $Y_{\tau_k^{(1)}}^{(i)}$, shows that also the first system has only finitely many B-particles at time $\tau_k^{(1)}$. Moreover, $\tau_{k+1}^{(1)}$ is the first time after $\tau_k^{(1)}$ at which some B-particle ρ coincides with some A-particle ρ' in the first system. From the right continuity of the paths $\pi_A(\cdot,\zeta)$ for all B-particles ζ plus (2.20), it then follows that $\tau_{k+1}^{(1)} > \tau_k^{(1)}$, a.s. By the induction hypothesis, ρ must also have type B at time $\tau_k^{(1)}$ in the second system. Therefore also ρ' turns into a B-particle in the second system no later than $\tau_{k+1}^{(1)}$. (ρ' may already have turned to type B before $\tau_{k+1}^{(1)}$ in the second system, in which case no type change occurs for ρ' in the second system at $\tau_{k+1}^{(1)}$). In any case, any particle which turns to type B at time $\tau_{k+1}^{(1)}$ in the first system also has type B at or before time $\tau_{k+1}^{(1)}$ in the second system, so that (5.2) with $\sigma^{(i)}$ replaced by $Y_{\tau_{k+1}^{(1)}}^{(i)}$ still holds. This completes the inductive step.

Since we already know that there are only finitely many B-particles at each $\tau_k^{(1)}$ in the second system, we conclude that a.s. this holds in both systems at each $\tau_k^{(1)}$. As we remarked above this shows that a.s. $\tau_{k+1}^{(1)} > \tau_k^{(1)}$ for all k, so that $\widehat{\tau} < \tau_\infty$ has probability 0 in both systems. Also, at each $\tau_k^{(1)}$, the number of B-particles in the first system is at most equal to the number of B-particles in the second system. Since there are at least k B-particles in the first system at time $\tau_k^{(1)}$, this shows that

$$P^{\sigma^{(1)}}\{\tau_\infty < \infty\} \le P^{\sigma^{(2)}}\{\tau_\infty < \infty\} = 0.$$

Thus, $\sigma^{(1)} \in \Sigma_0$. ■

Proof of Proposition 3. Fix K. Note that if a particle has type B at some time $s \le t$ and is outside the cube $\mathcal{C}(C_1 t)$ at that time, then by symmetry of the random walk $\{S\}$, the particle has a conditional probability, given $\mathcal{F}_s$, at least 1/2 of being outside $\mathcal{C}(C_1 t)$ at time t. Therefore

$$\begin{aligned}&E\{\text{number of particles outside } \mathcal{C}(C_1 t) \text{ at some time } s \le t\}\\ &\le 2E\{\text{number of particles outside } \mathcal{C}(C_1 t) \text{ at time } t\}.\end{aligned}$$

Thus, by (1.3)

$$P\{\text{a site outside } \mathcal{C}(C_1 t) \text{ is visited by a } B\text{-particle during } [0,t]\} \le t^{-K-1} \tag{5.3}$$

for $t \ge$ some t_0. We may therefore restrict ourselves for (1.7) to space-time points (z,t) with $z \in \mathcal{C}(C_1 t)$. Now fix a (z,t) which satisfies this condition and assume z is first visited by a B-particle at time $s \le t - [K_1 t \log t]^{1/2}$. Outside a set of probability $D_B t^{-K-d-1}$ this B-particle stayed at z for at least t^{-K-d-1} units of time, so that it is still at z at a time s of the form $kt^{-K-d-1} \le t - [K_1 t \log t]^{1/2} + t^{-K-d-1}$. Further, the probability that there is an A-particle at (z,t) which was at a point y with $\|y - z\| > [K_2 t \log t]^{1/2}$ at one of the times kt^{-K-d-1} is bounded by

$$\sum_{\substack{s = kt^{-K-d-1} \\ \le t - [K_1 t \log t]^{1/2} + 1}} \sum_{y: \|y-z\| > [K_2 t \log t]^{1/2}} EN_A(y,s) P\{S_{t-s} = z - y\}. \tag{5.4}$$

We now remind the reader of the particle system $\mathcal{P}^*$ which we introduced just before (2.17). In this process interactions between particles are ignored. Thus, in $\mathcal{P}^*$, each A-particle ρ with initial position $\pi(0,\rho)$ continues to follow the path $t \mapsto \pi(0,\rho) + \pi_A(t,\rho)$ even after the switching time $\theta(\rho)$. In the present case with $D_A = D_B$, this is the path which the particle follows anyway. We write $N^*(z,t)$ for the number of particles at (z,t) in $\mathcal{P}^*$. In our case this is just the total number of particles at (z,t) which are different from the finitely many original B-particles. $\{N^*(z,t) : z \in \mathbb{Z}^d, t \ge 0\}$ is stationary in time, and at each t the $N^*(x,t), x \in \mathbb{Z}^d$, are i.i.d. mean μ_A Poisson variables. From this description we see that

$$N_A(z,t) \le N^*(z,t), \text{ and } N_A(z,t) + N_B(z,t) \ge N^*(z,t) \quad z \in \mathbb{Z}^d, t \ge 0. \tag{5.5}$$

In particular, $EN_A(z,t) \le EN^*(z,t) = \mu_A$. Therefore, (5.4) is bounded by

$$\sum_{\substack{s=kt^{-K-d-1}\\ \le t-[K_1 t\log t]^{1/2}+1}} \mu_A P\{\|S_{t-s}\| > [K_2 t \log t]^{1/2}\} \le K_3 t^{K+d+2} \exp[-K_4 K_2 \log t] \tag{5.6}$$

(see (2.42) in [KS]). Taking into account the number of possibilities for z we find that the probability in the left hand side of (1.7) is for large t bounded by

$$\begin{aligned}&\frac{1}{t^{K+1}} + K_5 t^d \frac{1}{t^{K+d+1}} + K_6 t^{K+2d+2} \exp[-K_4 K_2 \log t]\\ &+ \sum_{\substack{s=kt^{-K-d-1}\\ \le t-[K_1 t\log t]^{1/2}+1}} \sum_{z \in \mathcal{C}(C_1 t)} \sum_{y:\|y-z\| \le [K_2 t \log t]^{1/2}} P\{\text{there is a } B\text{-particle}\\ &\qquad \text{at } z \text{ and an } A\text{-particle at } y \text{ at time } s = kt^{-K-d-1}, \text{ but this,}\\ &\qquad A\text{-particle is not turned into a } B\text{-particle by time } t\}.\end{aligned}$$

We shall write $\mathcal{U}(k,z,y)$ for the event in the summand corresponding to k,z,y in the triple sum here. We further write s for kt^{-K-d-1}. We choose K_2 so large that for $t \ge$ some t_1 the sum of the first three terms here is at most $1/(2t^K)$. The probability in the left hand side of (1.7) is then bounded by

$$\frac{1}{2t^K} + \sum_{\substack{s=kt^{-K-d-1}\\ \le t-[K_1 t\log t]^{1/2}+1}} \sum_{z \in \mathcal{C}(C_1 t)} \sum_{y:\|y-z\| \le [K_2 t \log t]^{1/2}} P\{\mathcal{U}(k,z,y)\}. \tag{5.7}$$

The triple sum in (5.7) contains at most $K_7 t^{K+d+2}[C_1 t]^d\,[K_2 t \log t]^{d/2} \le K_8 t^{K+3d+2}$ summands. We shall complete the proof of (1.7) by showing that each summand in (5.7) is at most $K_9 t^{-2K-3d-3}$ for some constant K_9 which does not depend on t.

At an intuitive level it seems clear that there should be a way to estimate these summands which is similar to the one used in the proof of Theorem 2 for estimating the probability that a fixed vertex x has not been reached by a B-particle by time t. We have to be a bit careful, though, not to bias the relevant distributions by the fact that there is a B-particle at the space-time point (z,s) or a certain number of A-particles at (y,s). Nevertheless, the proof will follow Theorem 2 closely. As before, we shall write $\mathcal{P}_0$ for our original particle system.

In order to estimate the summand in (5.7) we shall make use of the monotonicity property in Lemma 15. We shall compare the real system with a modified system, which is constructed via two modifications. In each of these modifications we remove particles and change some B-particles to A-particles, at time s. According to the monotonicity property of Lemma 15, any particle which is present in both systems at time t, and which has type A in the unmodified system, also must have type A in the modified system. Now fix y and z as well as $s < t$, and make the first modification as follows: at time s, at each $x \ne z$ remove all particles of type B which were added at time 0, and reset the type of the remaining particles at x to A.

At $x = z$, if there are particles at (z, s) give them type B at time s. If there is no particle at (z, s), put one B-particle at z at time s. If there is no B-particle at z or no A-particle at y at time s before the modification, then the k, z, y term in (5.7) is zero, so we do not care what the modification does in this case. In the other cases there is at least one A-particle at y, and a B-particle at z, so $y \neq z$. In these cases the A-particle at y is not removed, and the type of the particles at z is unchanged by the modification, so the monotonicity property gives us that $P\{\mathcal{U}(k, z, y)\}$ can only go up by this first modification. Note that after the first modification, we have $N^*(x, s)$ A-particles at x, for any $x \neq z$, and $N_B(z, s) \vee 1$ B-particles at z. These

$$N^*(x, s), x \in \mathbb{Z}^d, \text{ are i.i.d. mean } \mu_A \text{ Poisson variables.} \tag{5.8}$$

Now there are $N^*(y, s)$ A-particles at (y, s) after the first modification. Since all particles at the same space-time point play the same role, each of these $N^*(y, s)$ particles at (y, s) has the same probability of still having type A at time t. Order the particles at (y, s) by some arbitrary rule. Then the k, z, y summand in (5.7) is at most

$$E\Big\{N^*(y, s)P\{\text{the first particle at } (y, s) \text{ is still of type } A \\ \text{at time } t \text{ in the first modified system} | \mathcal{F}_s\}\Big\}. \tag{5.9}$$

There is some dependence between $N^*(y, s)$ and the conditional probability factor in this expectation. To handle this we make one further modification at y. In this second modification we remove all but the first A-particle at (y, s). If there is no A-particle at (y, s), then we add an A-particle at y. Since there is no contribution to (5.9) from the sample points with $N^*(y, s) = 0$, we can ignore how this modification influences the conditional probability in (5.9) on the event $\{N^*(s, y) = 0\}$. On the event $\{N^*(y, s) \geq 1\}$ the second modification cannot decrease the conditional probability in (5.9), once again by Lemma 15. Now, the second modified system does not involve $N^*(y, s)$ anymore. In the second modified system the conditional probability in (5.9) is replaced by

$$P\{\text{the unique particle at } (y, s) \text{ is still of type } A \\ \text{at time } t \text{ in the second modified system} | \mathcal{F}_s\}, \tag{5.10}$$

which is a function of the $N^*(u, s)$ with $u \neq y$ only. As we already observed, these are independent mean μ_A Poisson variables, independent of $N^*(y, s)$. Consequently (5.9) is at most

$$E\{N^*(y, s)\}P\{\text{the unique particle at } (y, s) \text{ is still of type } A \\ \text{at time } t \text{ in the second modified system}\}. \tag{5.11}$$

The statement (5.8) suggests that we introduce two systems $\mathcal{P}^{z,y}$ and $\mathcal{P}^z$ say, which have initial distributions as follows: Let $\widetilde{N}(u)$, $u \in \mathbb{Z}^d$, be a family of independent mean μ_A Poisson variables. Then $\mathcal{P}^{z,y}$ starts with $\widetilde{N}(u)$ A-particles at u if $u \notin \{z,y\}$, $\widetilde{N}(z) \vee 1$ B-particles at z and one A-particle at y. No other particles are in the initial state. For $\mathcal{P}^z$ the only change is that at y we put initially $\widetilde{N}(y)$ particles, so that y is treated like all other sites $u \neq z$. The particles then move and change type in $\mathcal{P}^{z,y}$ and in $\mathcal{P}^z$ according to the same rules as in $\mathcal{P}_0$. It follows from (5.8) that the probability factor in (5.11) equals

$$\begin{aligned} &P\{\text{the unique particle at } (y,0) \text{ is still of type } A \text{ at time } t-s \\ &\qquad\qquad \text{in the system } \mathcal{P}^{z,y}\} \\ &= P\{\text{the unique particle at } (y,0) \text{ is still of type } A \text{ at time } t-s \\ &\qquad\qquad \text{in the system } \mathcal{P}^z | \text{start with 1 } A\text{-particle at } y\} \\ &\leq \frac{1}{e^{-\mu_A}\mu_A} P\{\text{the first particle at } (y,0) \text{ is still of type } A \text{ at time } t-s \\ &\qquad\qquad \text{in the system } \mathcal{P}^z\}. \end{aligned} \tag{5.12}$$

Here $e^{-\mu_A}\mu_A$ in the right hand side represents $P\{\widetilde{N}(y) = 1\}$. We further have $E\{N^*(y,s)\} = \mu_A$, and $t - s \geq [K_1 t \log t]^{1/2} - 1$, so that the k, z, y summand in (5.7) is bounded by

$$\begin{aligned} &e^{\mu_A} P\{\text{the first particle at } (y,0) \text{ is still of type } A \\ &\qquad\qquad \text{at time } [K_1 t \log t]^{1/2} - 1 \text{ in the system } \mathcal{P}^z\}. \end{aligned} \tag{5.13}$$

Our task has now been reduced to estimating the probability factor in (5.13). But the system $\mathcal{P}^z$ is either equal to the original system $\mathcal{P}_0$ with one B-particle added at z (in case $\widetilde{N}(z) = 0$) or is like the system $\mathcal{P}_0$, but with all particles at z turned into B-particles at time 0, without the addition of an extra B-particle. In both these cases we can basically repeat the proof of Theorem 2 to estimate (5.13). We form a path $s \mapsto \lambda(s) = \lambda(s,y,z), s \geq 0$, which starts at $\lambda(0,y,z) = z$ and from there proceeds according to the rules (i)-(v) in the proof of Theorem 2 (after the proof of Proposition 9) with only one change in rule (v). We now want the path to have a drift to the first particle which started at y, instead of to a fixed vertex x. If we denote this first particle at y by ϕ, then the position of ϕ at a time s equals $\pi(s,\phi) = y + \pi_A(s,\phi)$. Accordingly we change rule (v) to the following:

(v') if $\widehat{\rho}(s-)$ jumps from $\lambda(s-) = w$ to w' at some time s such that there is at least one other particle ρ' at w at time $s-$, then $\lambda(\cdot)$ jumps to w' at time s if and only if $\|w' - \pi(s,\phi)\|_2 < \|w - \pi(s,\phi)\|_2$, and in this case again $\widehat{\rho}(s) = \widehat{\rho}(s-)$; if, however, $\|w' - \pi(s,\phi)\|_2 \geq \|w - \pi(s,\phi)\|_2$, then $\lambda(\cdot)$ does not jump at time s and we take $\widehat{\rho}(s) = \rho'$.

Under these changed rules $\lambda(\cdot) - \pi(\cdot, \phi)$ has a drift towards zero in the sense that now

$$\begin{aligned}\widetilde{M}(t) &:= \|\lambda(t,x) - \pi(t,\phi)\|_2 \\ &\quad - D_A \int_0^t \big[I_1(u)\widetilde{\Gamma}_1(u) + I_{\geq 2}(u)\widetilde{\Gamma}_{\geq 2}(u)\big]du - D_A \int_0^t \widetilde{\Gamma}_1(u)du\end{aligned} \tag{5.14}$$

is an $\{\mathcal{F}_t\}$-martingale, where analogously to (4.52)

$$\begin{aligned}\widetilde{\Gamma}_1(u) &:= \frac{1}{2d}\sum_{i=1}^{2d} \big[\|\lambda(u) + e_i - \pi(u,\phi)\|_2 - \|\lambda(u) - \pi(u,\phi)\|_2\big], \\ \widetilde{\Gamma}_{\geq 2}(u) &:= \frac{1}{2d}\sum{}^* \big[\|\lambda(u) + e_i - \pi(u,\phi)\|_2 - \|\lambda(u) - \pi(u,\phi)\|_2\big],\end{aligned}$$

and $\sum{}^*$ is the sum over those $i \in \{1, \dots, 2d\}$ for which

$$\|\lambda(u) + e_i - \pi(u,\phi)\|_2 - \|\lambda(u) - \pi(u,\phi)\|_2 < 0,$$

and $e_{d+i} = -e_i, 1 \leq i \leq d$; $I_1(u)$ and $I_{\geq 2}(u)$ are the same as in (4.51). The extra integral $D_A \int_{[0,t]} \widetilde{\Gamma}_1(u)du$ (which was not present in the $M(\cdot)$ of (4.53)) has to be introduced to compensate for the jumps of ϕ. However, the proof that $\widetilde{M}$ is a martingale is quite the same as for Lemma 10.

From here on one can follow the proof of Theorem 2. One merely has to replace $\lambda(s) - x$ by $\lambda(s) - \pi(s, \phi)$ at most places and to take into account the jumps in $\widetilde{M}(\cdot)$ due to a jump of ϕ. For instance the definition of σ_{k+1} right after (4.60) should now be

$$\begin{aligned}\sigma_{k+1} = \min \big[\sigma_k + 1, \inf\{t > \sigma_k :\ & t \text{ is a jump of the} \\ & \text{distinguished partcle } \widehat{\rho} \text{ or of } \phi\}\big].\end{aligned}$$

One also has to note that t should be replaced by $[K_1 t \log t]^{1/2} - 1$ in everything that comes after Lemma 11 in Section 4, but this is a trivial change to make. The conclusion is that if K_1 is chosen sufficiently large with respect to K_2, then each of the summands in (5.7) is bounded by $K_7 t^{-3d-2K-3}$, uniformly in k, y, z in the ranges over which they are summed. As pointed out before this completes the proof of (1.7).

Once we have (1.7), (1.8) easily follows by means of (1.5). Indeed, by (1.5), outside a set of probability at most t^{-K}, each point in $\mathcal{C}\big((C_2/2)t\big)$ has been visited by a B-particle during $[0, t/2]$. We then have by (1.7) that the additional probability of some vertex $z \in \mathcal{C}\big((C_2/2)t\big)$ being occupied by an A-particle at time t is at most t^{-K}. Thus, for large t the left hand side of (1.8) is at most $2t^{-K}$. ■

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